Zero-Knowledge Proofs And ZK-SNARKs (2): Concrete Protocols

Foundations Seminar

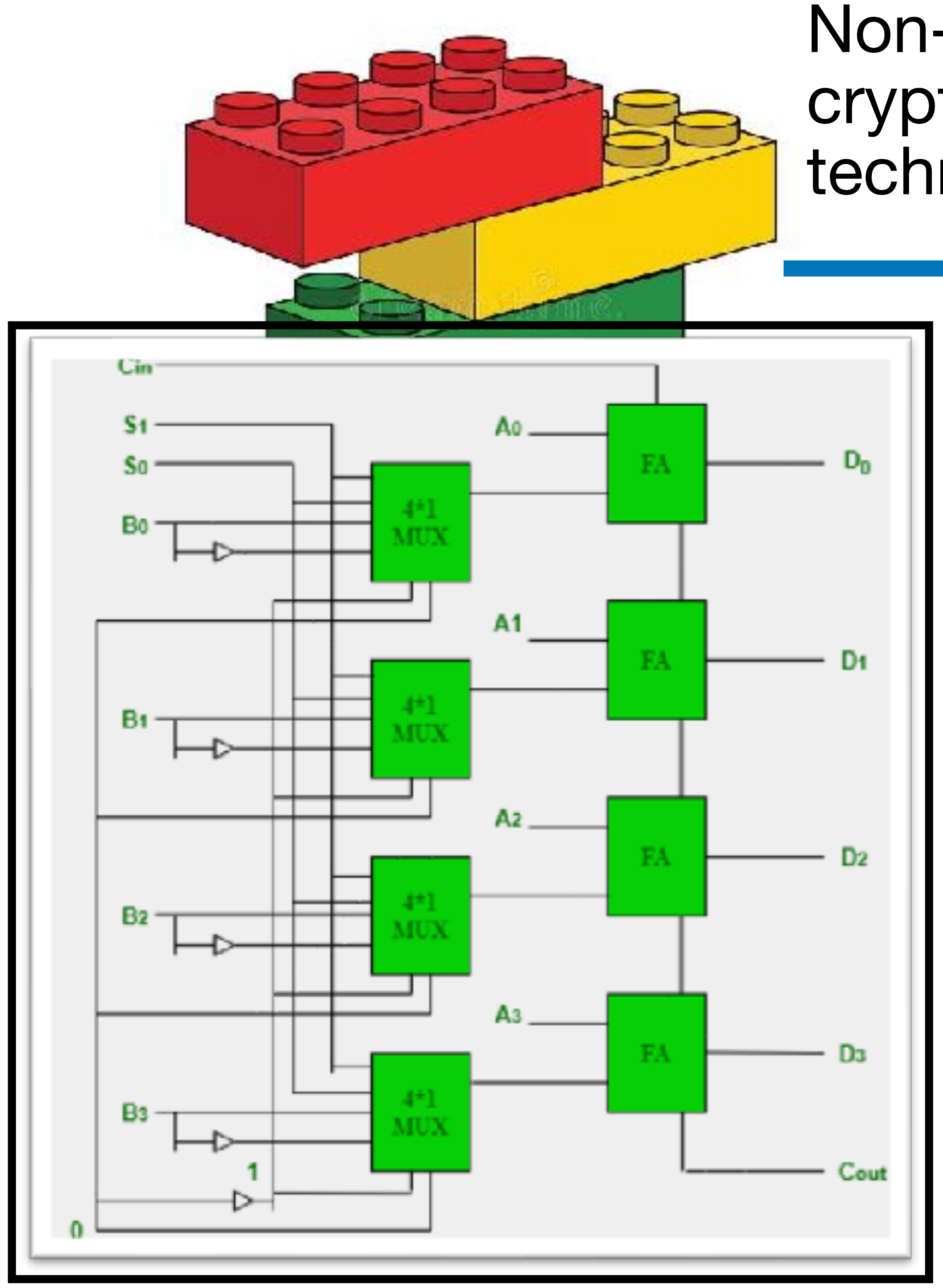
Helger Lipmaa, April 8, 2025

Up To Now

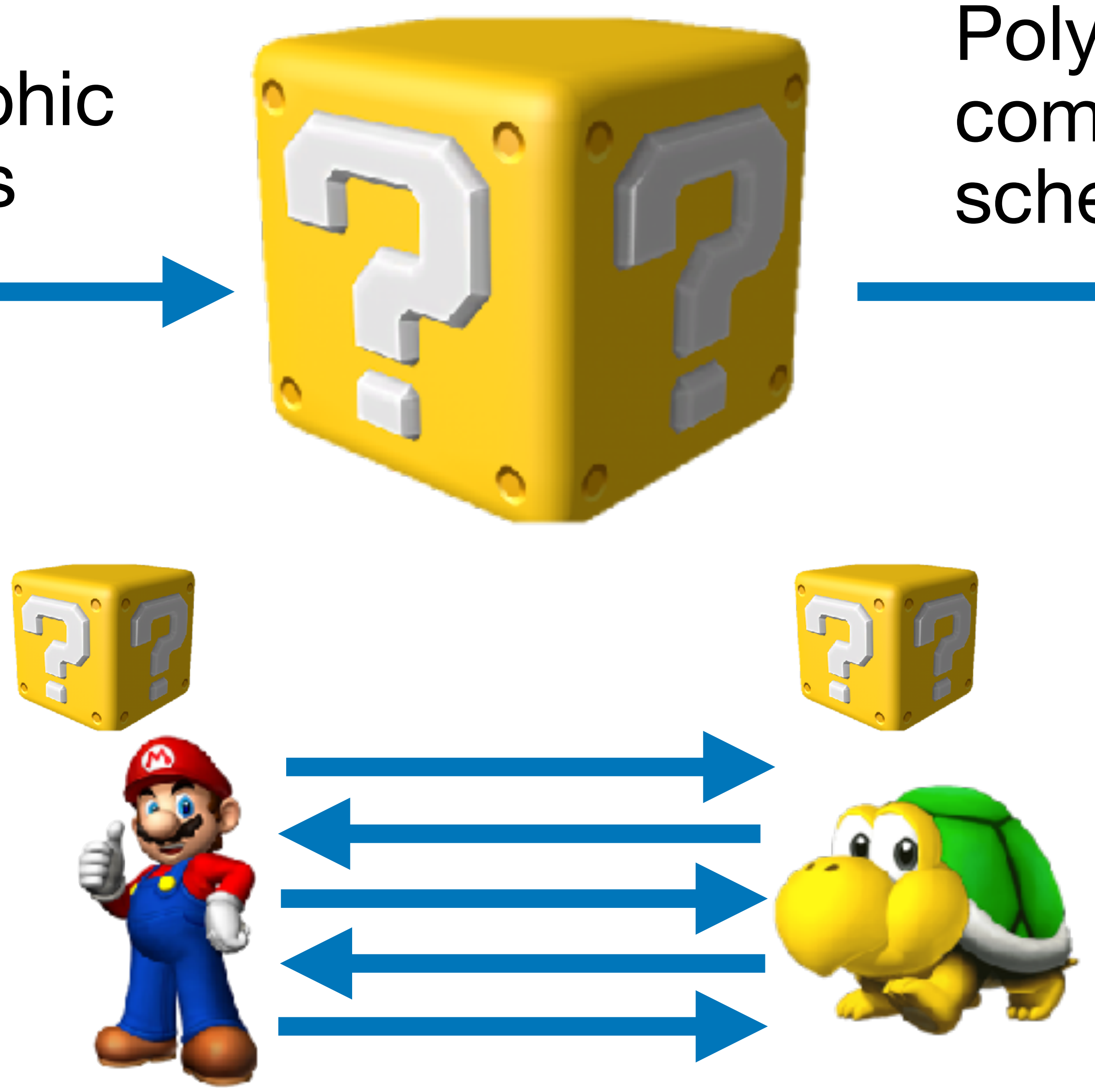
Intermediate Representation

(Polynomial) Interactive Oracle Proof

ZK-SNARKs



Non-cryptographic techniques



Polynomial commitment scheme

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- We explored the current high-level landscape of zk-SNARKs

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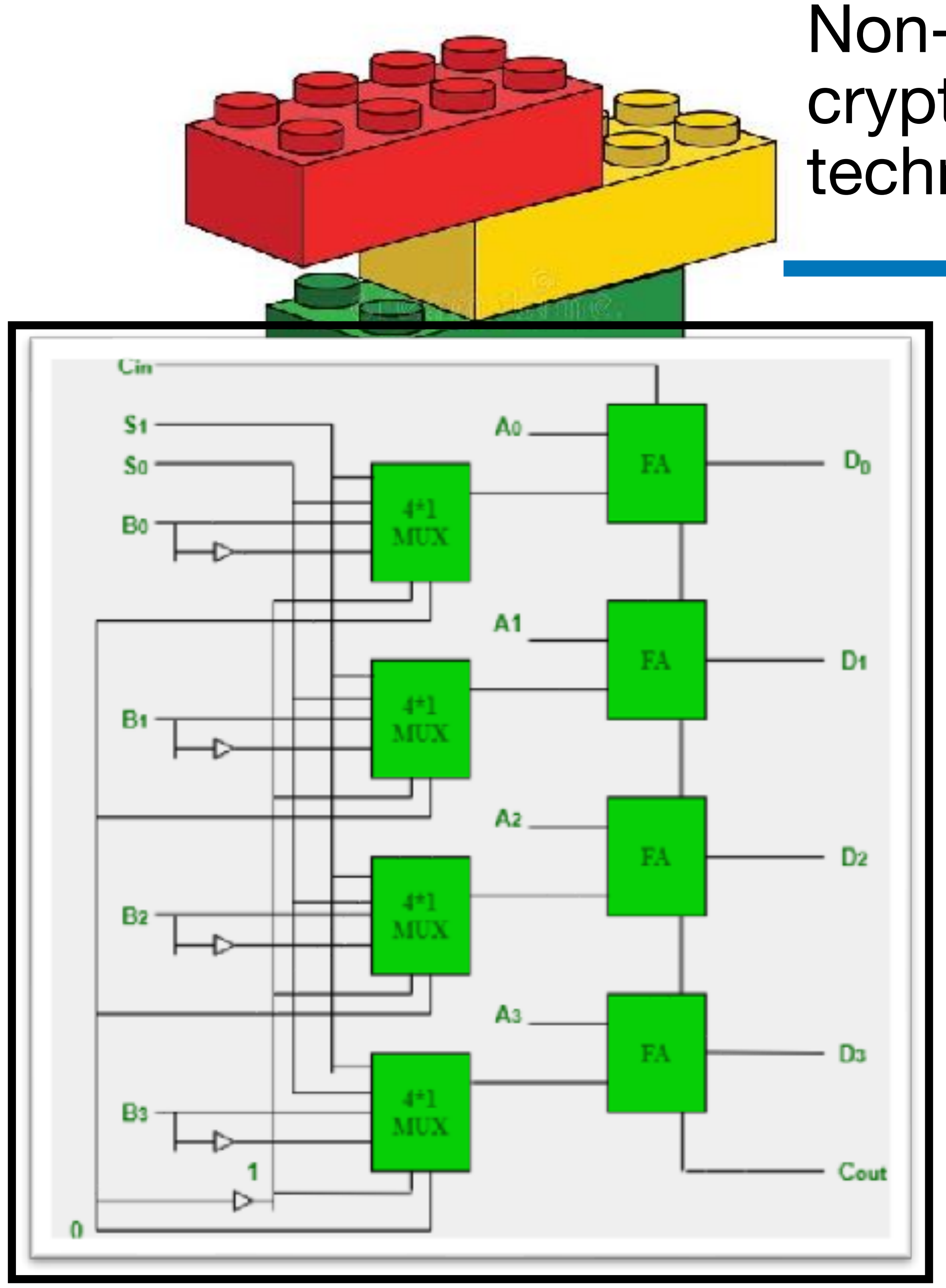

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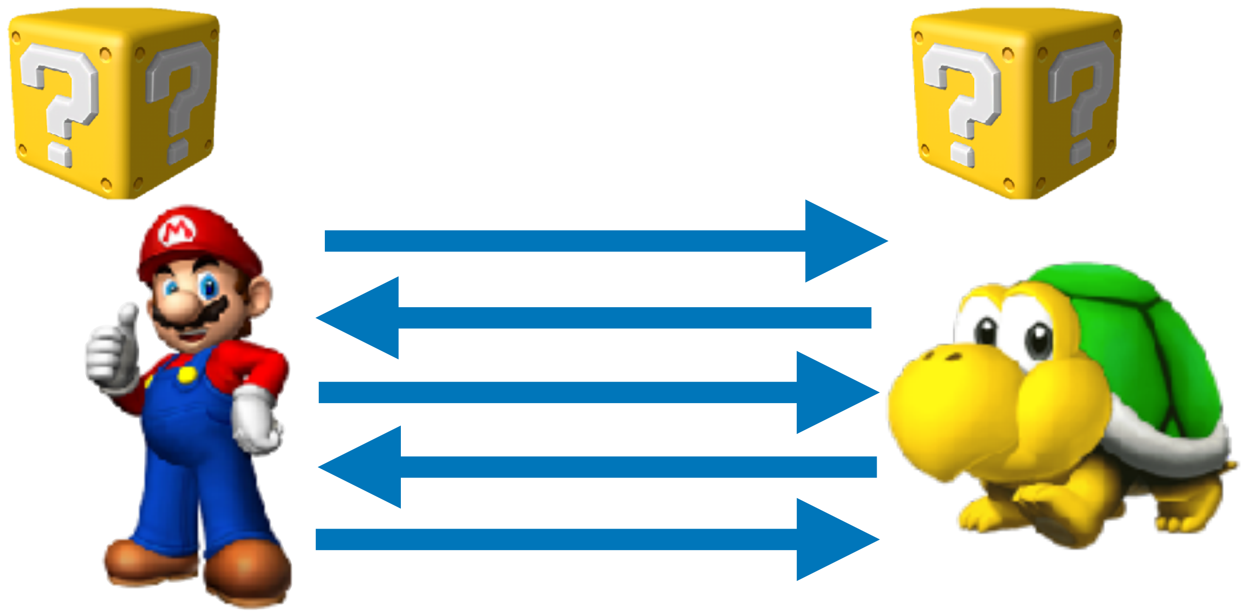

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- Input size $n \geq 2^{24}$, companies are pushing for $n \geq 2^{28}$

Reminder: FFT (NTT)

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  - => almost all univariate PIOP based SNARKs use such fields

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  - We will explain that next…

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  - Assuming the oracle guarantees the polynomial has "low degree" $\leq n - 1$

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Why does this idea work?

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- **Lemma (Schwartz-Zippel).** Let $f(X) \in \mathbb{F}[X_1, \ldots, X_m]$ be a **non-zero** polynomial of total degree $n \geq 0$. Let S be a finite subset of $\mathbb{F}$. Let $r_1, \ldots, r_m \leftarrow_{\$} S$ be sampled uniformly at random. Then the probability that $f(r_1, \ldots, r_m) = 0$ is at most $n/|S|$.

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- **Lemma.** Let $f(X) \in \mathbb{F}_{\leq n}[X]$ be a **non-zero** polynomial of degree $n \geq 0$. Let $r \leftarrow_{\$} \mathbb{F}$ be sampled uniformly at random. The probability that $f(r) = 0$ is at most $n/|\mathbb{F}|$
- **Proof:** Straightforward since $f(X)$ has at most n roots
- Schwartz-Zippel lemma generalises this to multivariate poly-s and subsets of $\mathbb{F}$
- **Lemma (Schwartz-Zippel).** Let $f(X) \in \mathbb{F}[X_1, \ldots, X_m]$ be a **non-zero** polynomial of total degree $n \geq 0$. Let S be a finite subset of $\mathbb{F}$. Let $r_1, \ldots, r_m \leftarrow_{\$} S$ be sampled uniformly at random. Then the probability that $f(r_1, \ldots, r_m) = 0$ is at most $n/|S|$.
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- Schwartz-Zippel is hugely important in constructing efficient zk-SNARKs
- We mostly just use the first lemma (but still call it Schwartz-Zippel)

On Schwartz-Zippel

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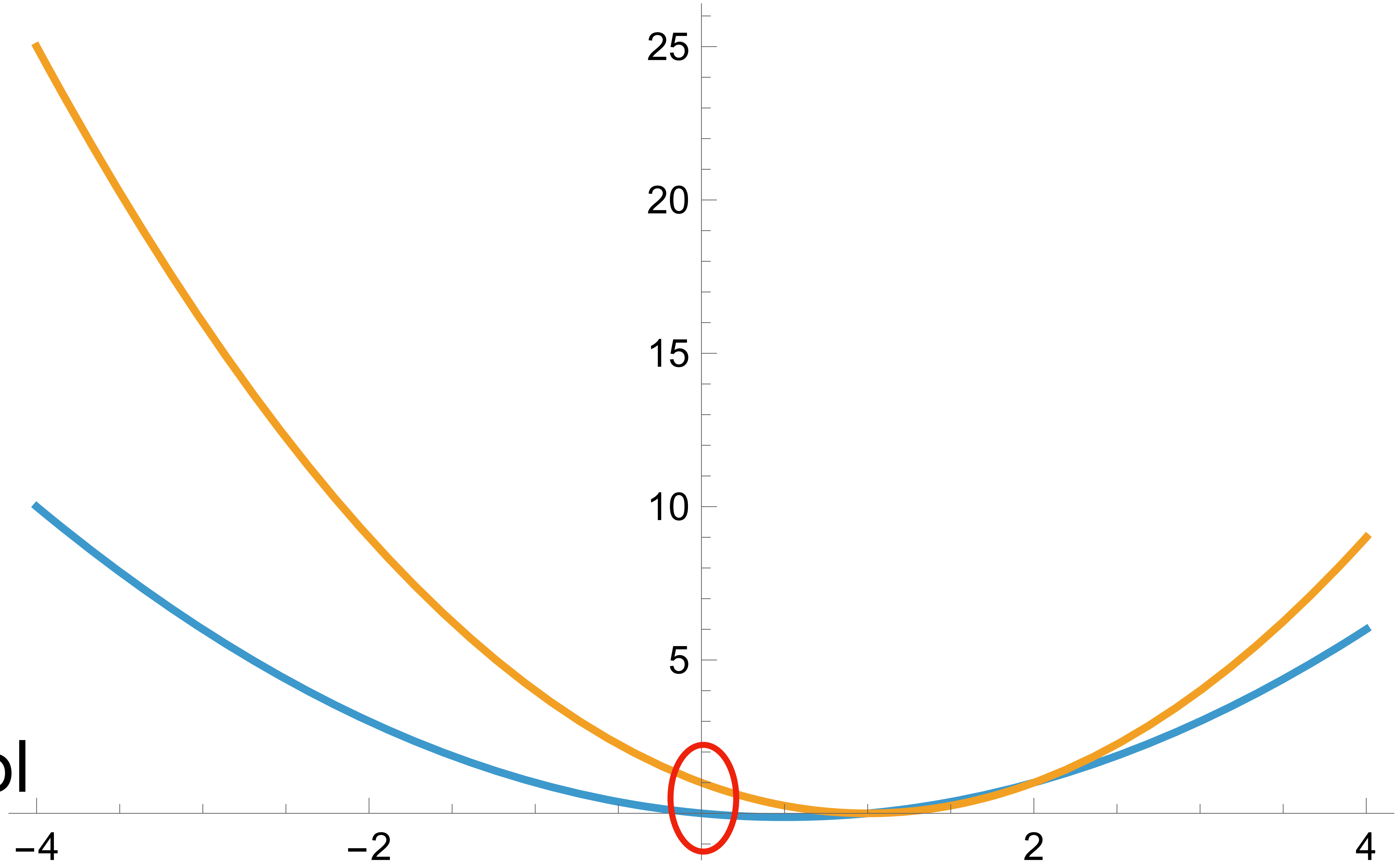

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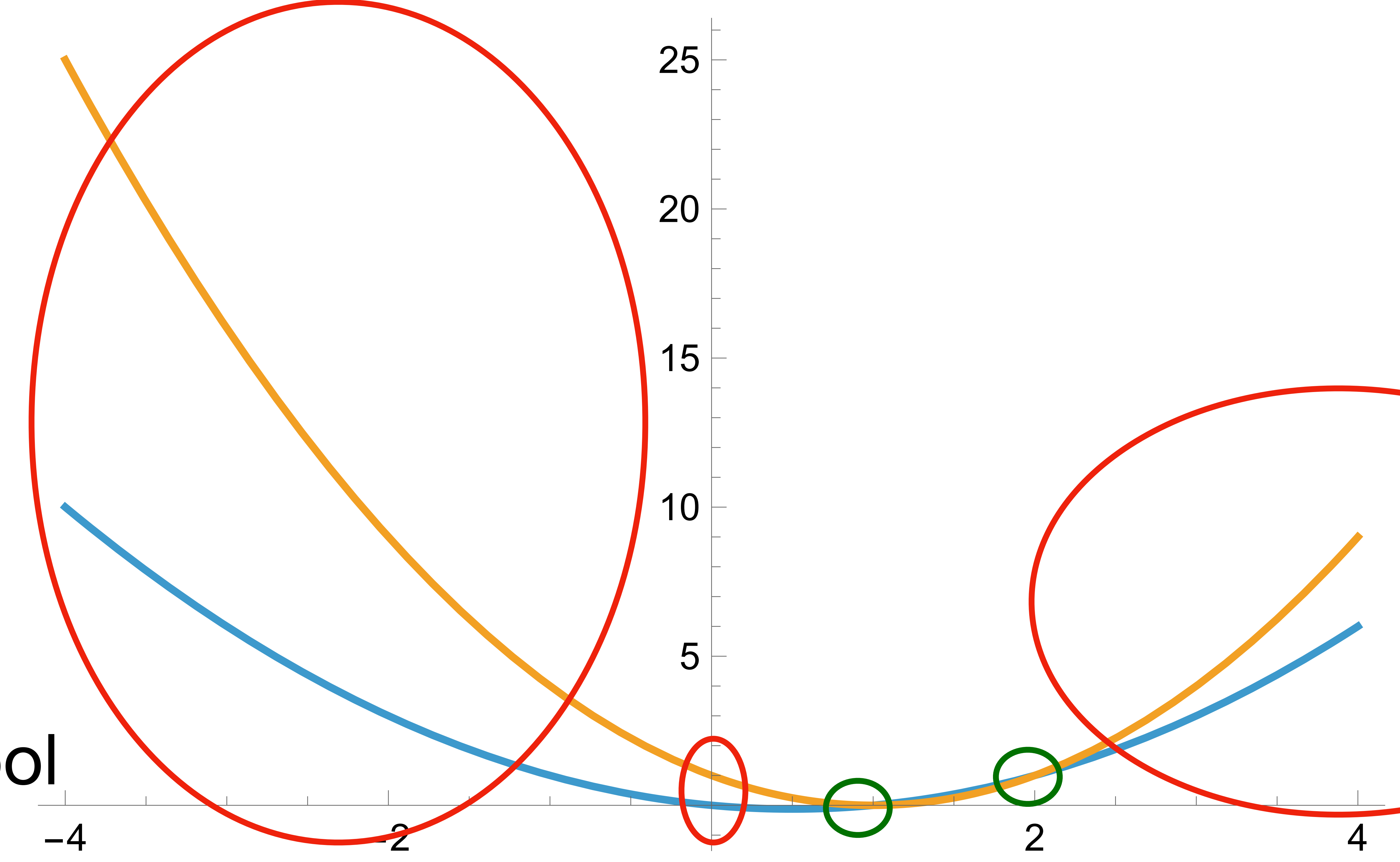

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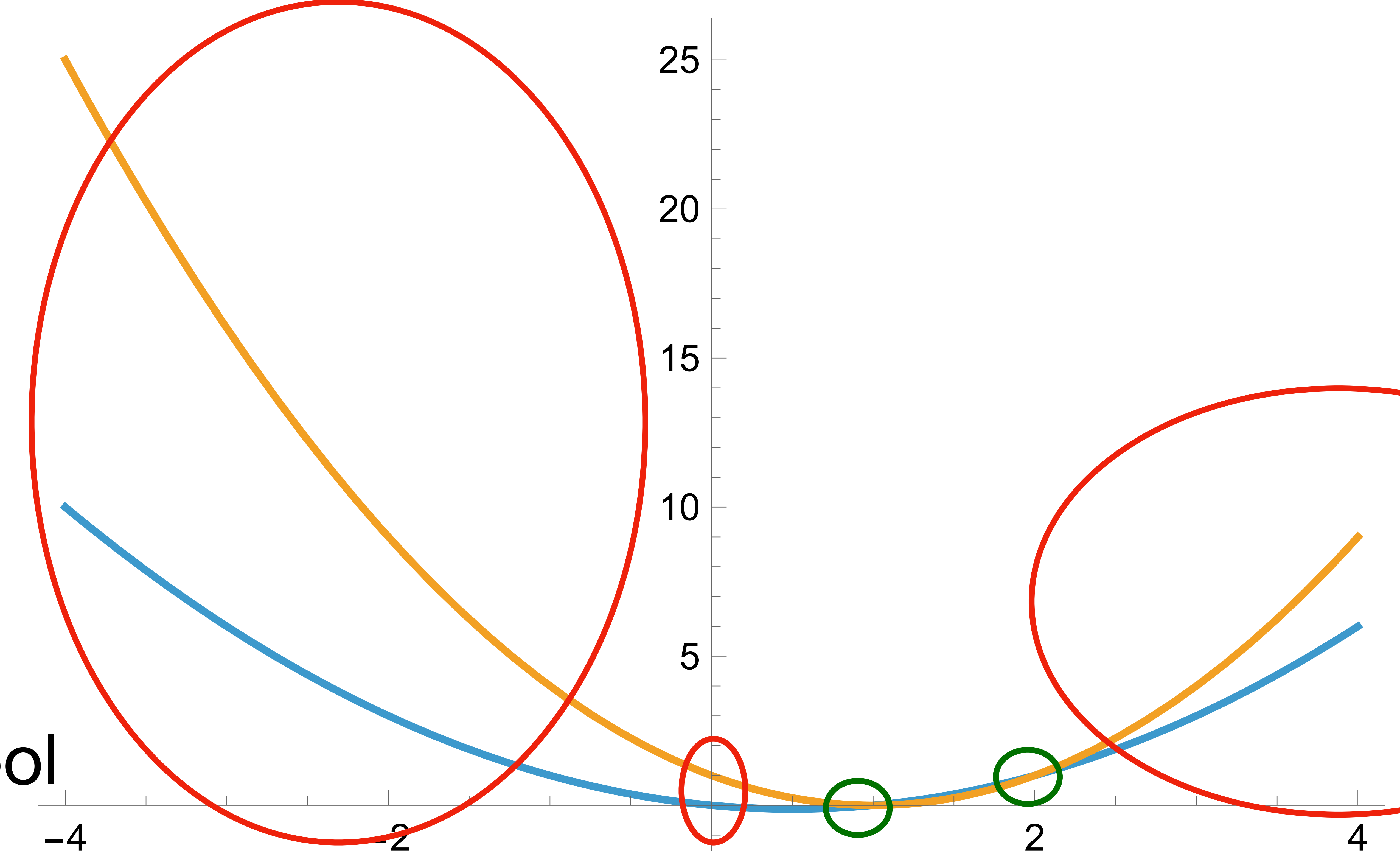

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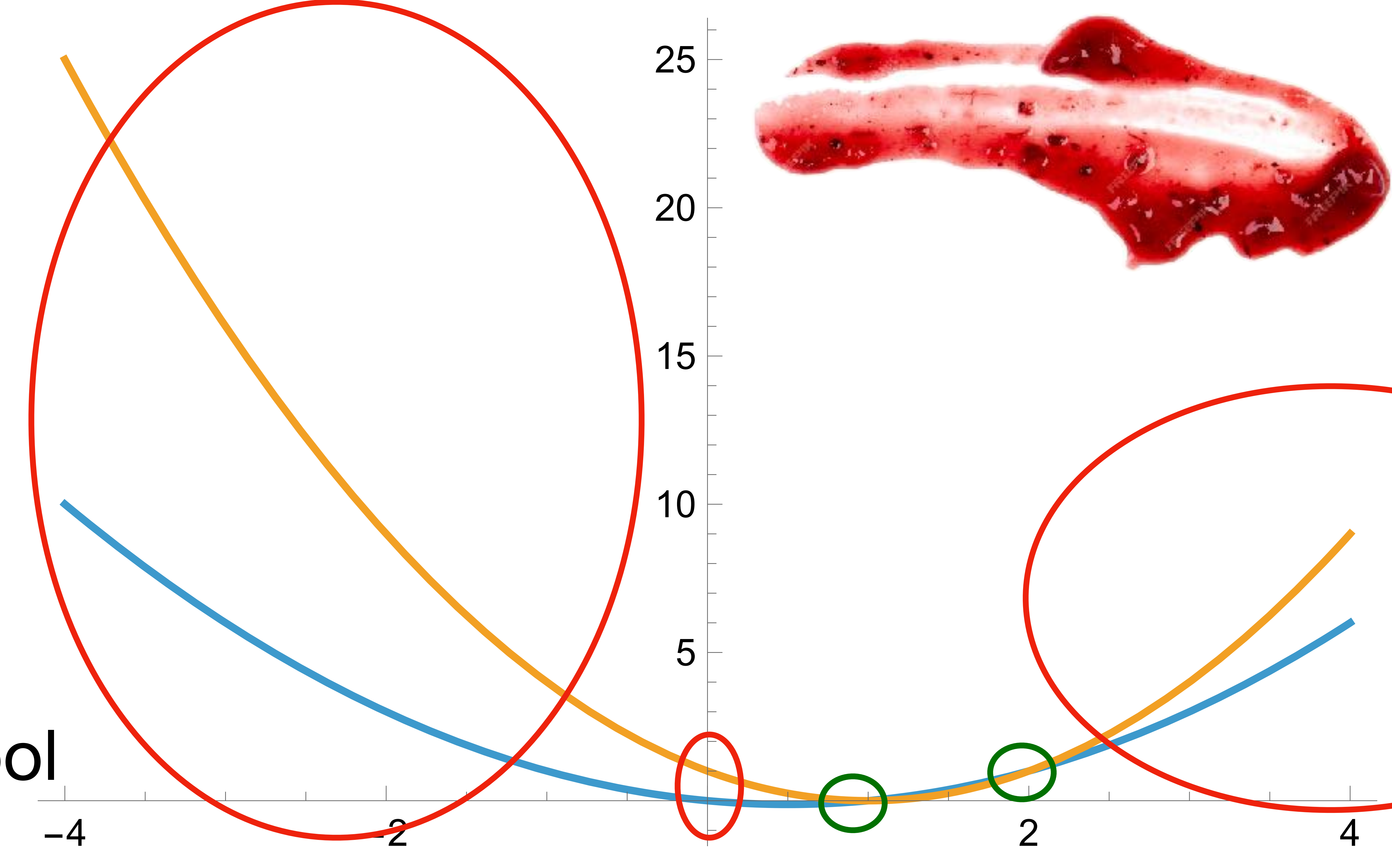

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  - “Smears" around the error — akin to error-correcting codes

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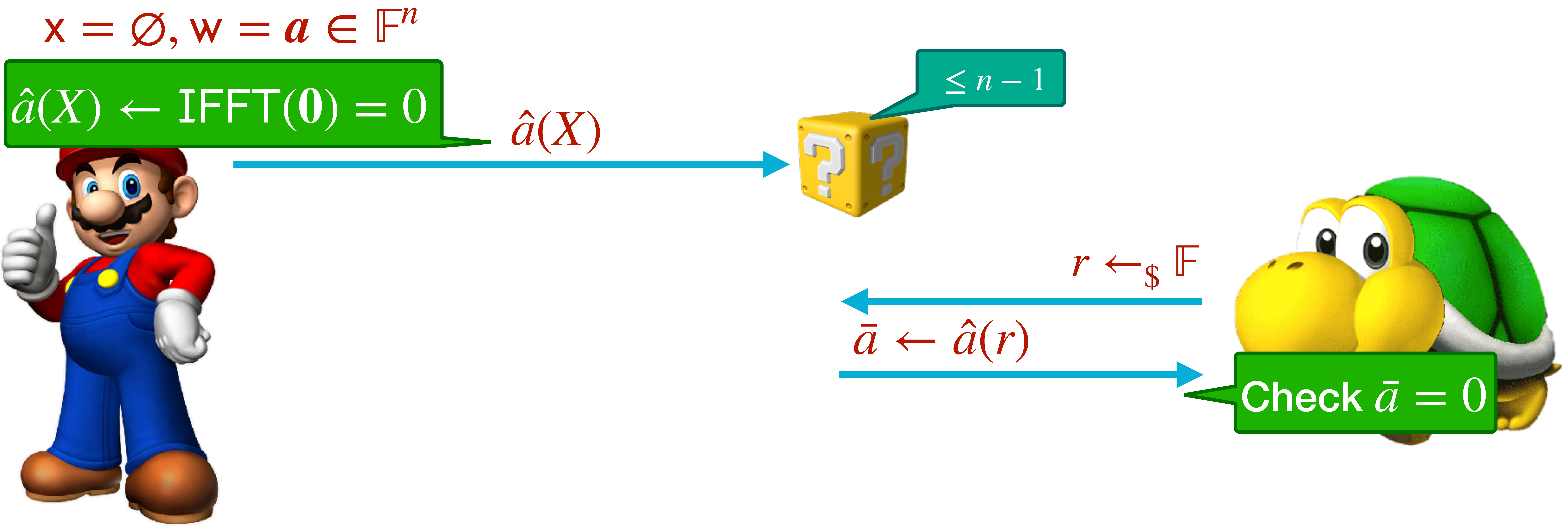

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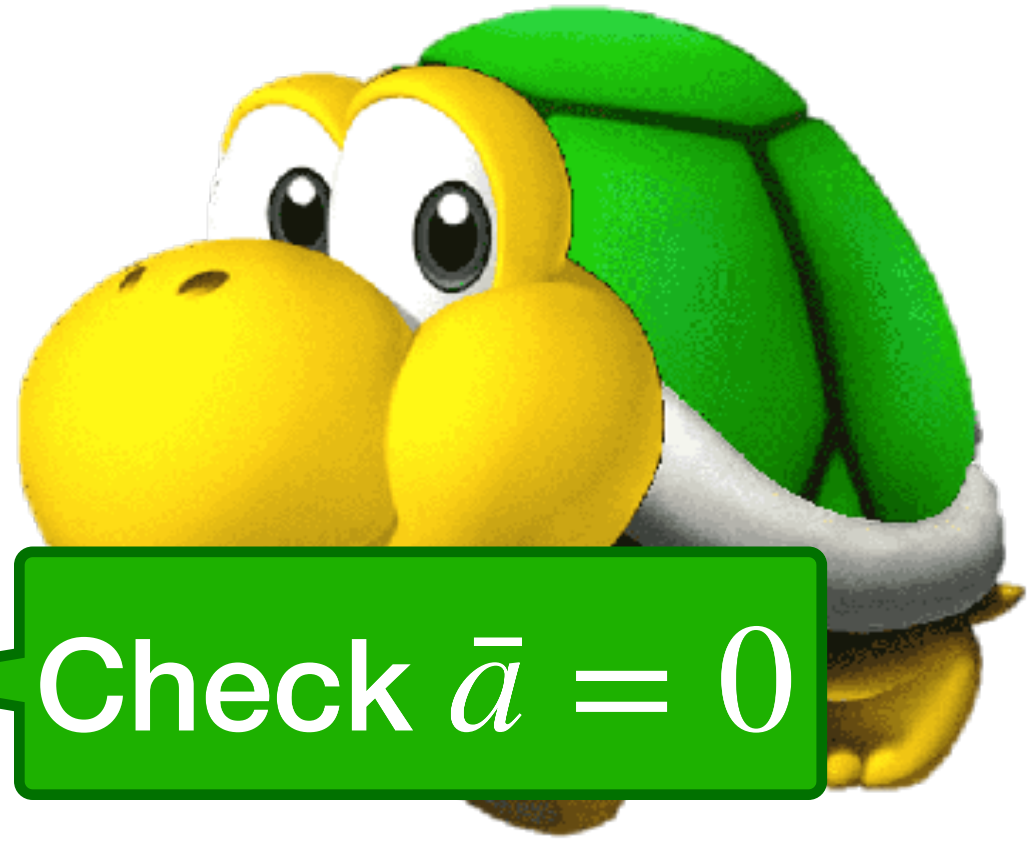

Check $\bar{a} = 0$

- Note: the goal of the protocol is to check $\boldsymbol{a} = \mathbf{0}$
- This protocol makes little sense if nothing about $\boldsymbol{a}$ is given as an input to the protocol // what exactly is $= \mathbf{0}$?
- Solution: an oracle $[\![\hat{a}(X)]\!]$ is a part of the input
- $\mathscr{R} = \{(\mathrm{x}, \mathrm{w}) : \mathrm{x} = [\![\hat{a}(X)]\!] \wedge \mathrm{w} = \mathrm{FFT}(\hat{a}(X)) \wedge \mathrm{w} = \mathbf{0}\}$

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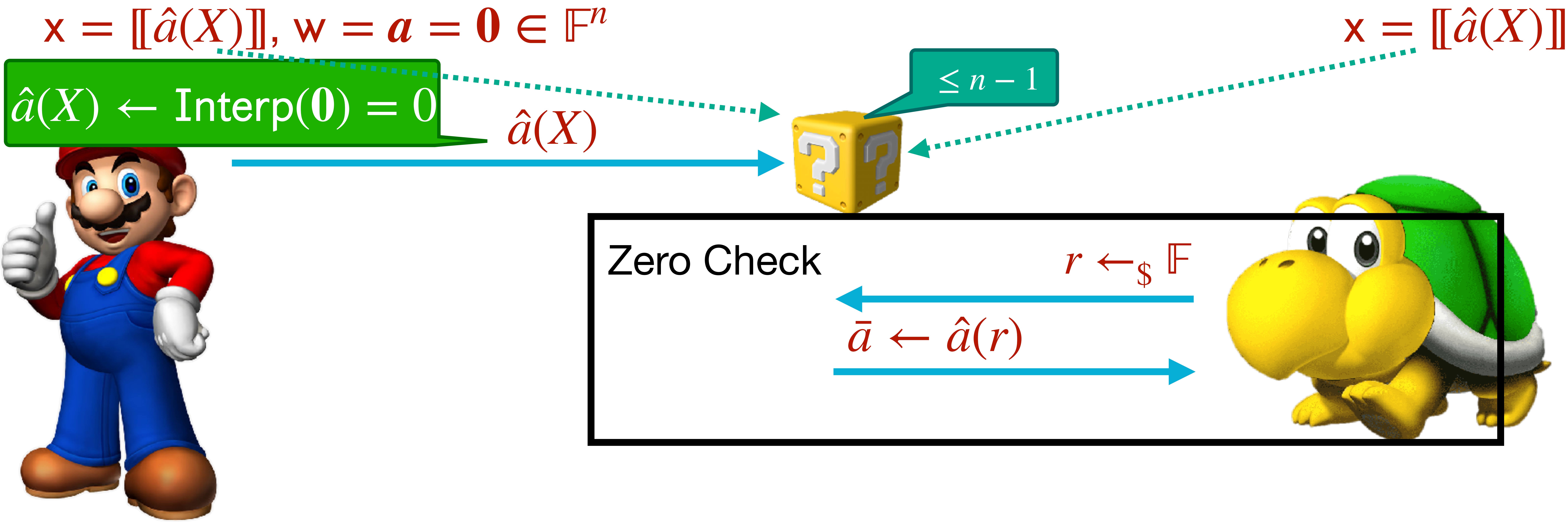

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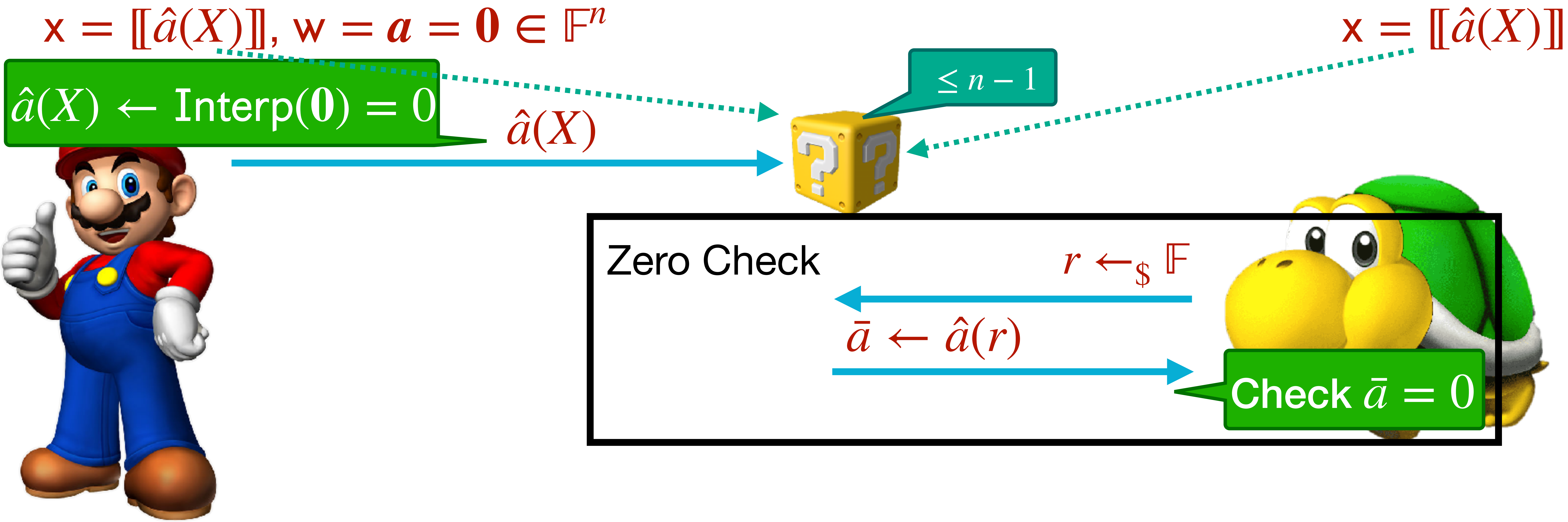

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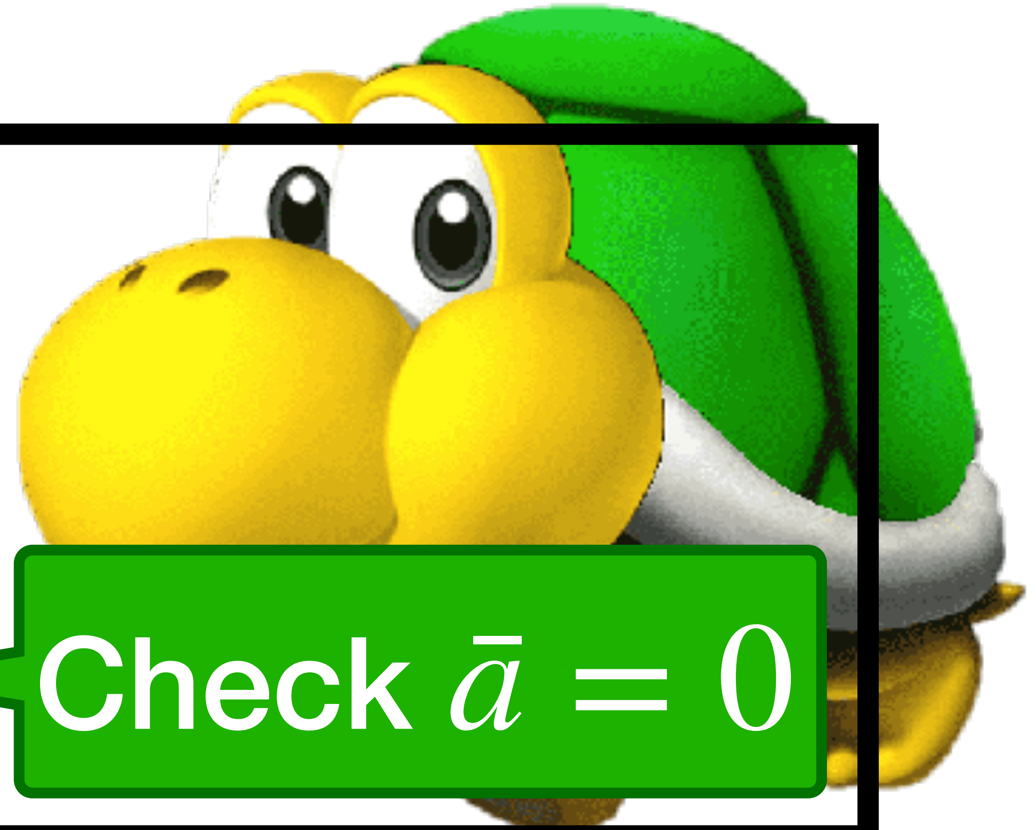

Zero Check

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- In general, PIOP is a **proof of knowledge** of **knowing** the **contents of the oracles** that satisfy some relation
- In zk-SNARKs, when replacing oracles with commitments, we get a proof of knowledge of knowing the contents of the **commitments** that satisfy some relation

Product Check

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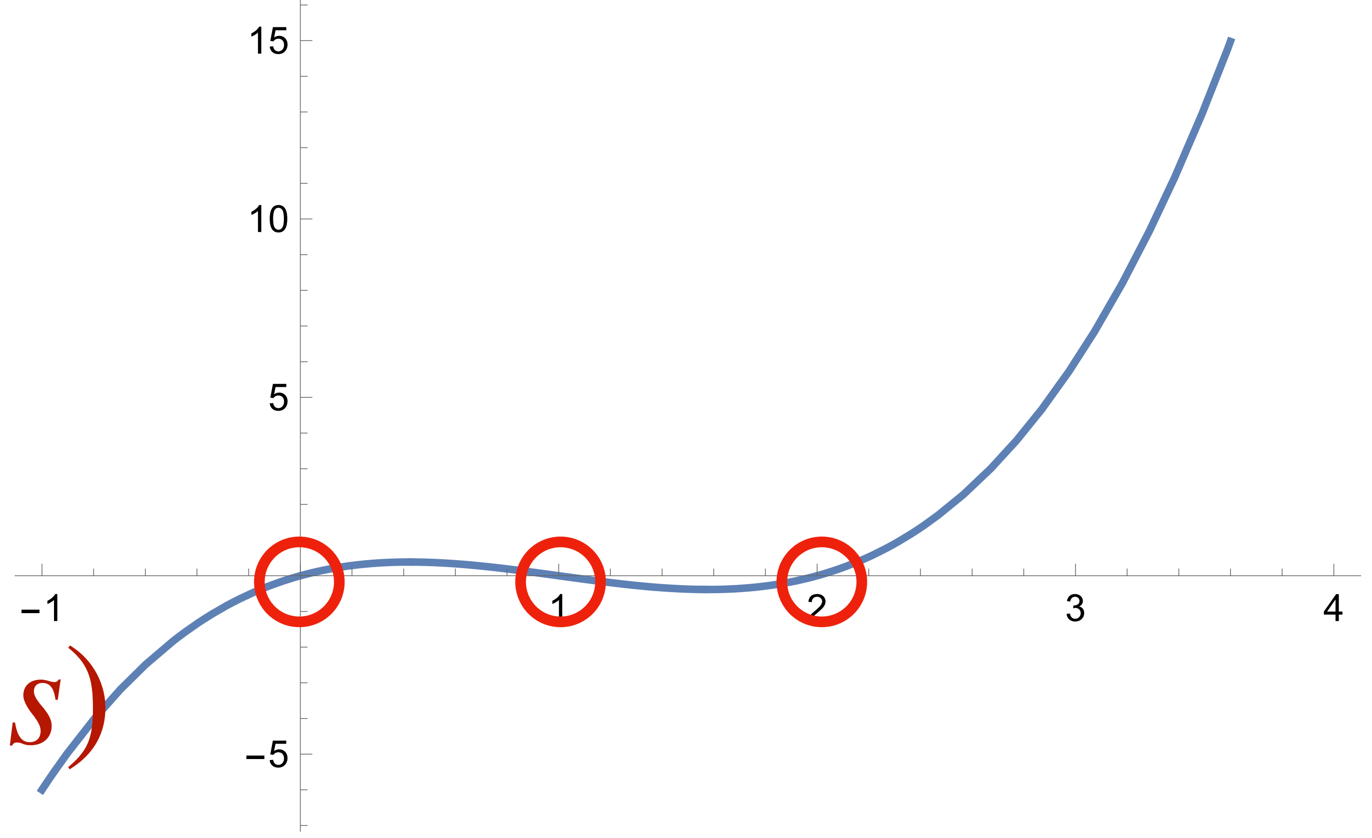
- In addition, adding ZK will increase the degree of “virtual" oracles

Vanishing Polynomials



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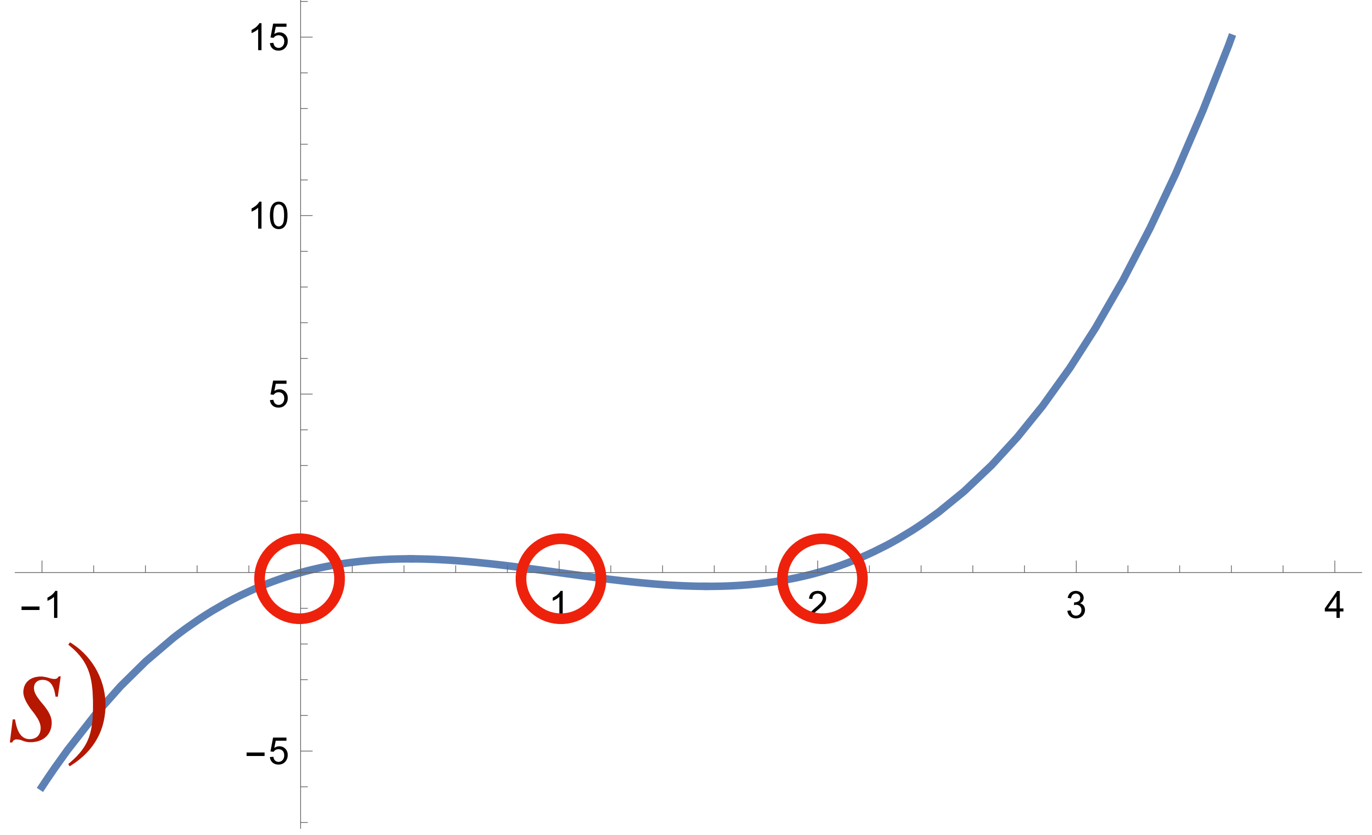

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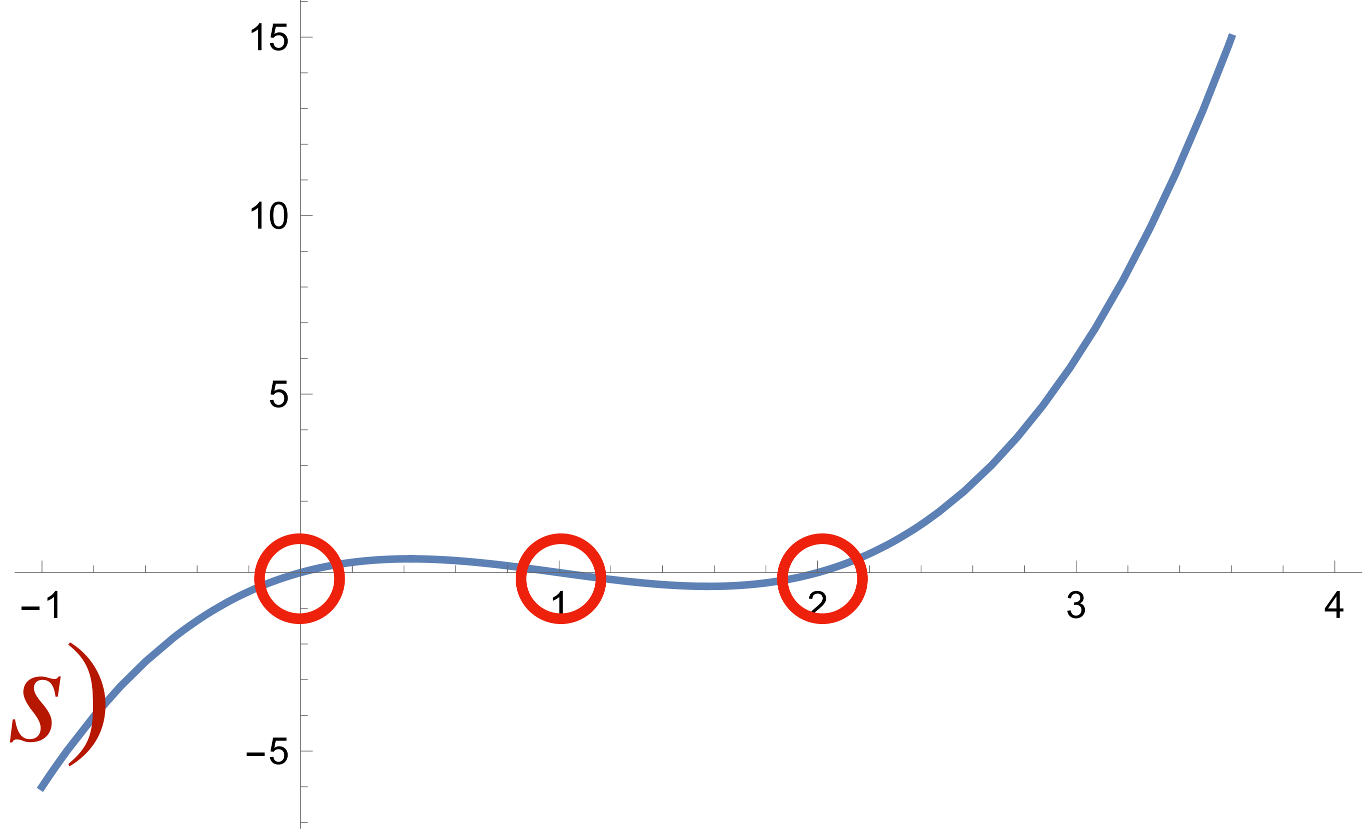

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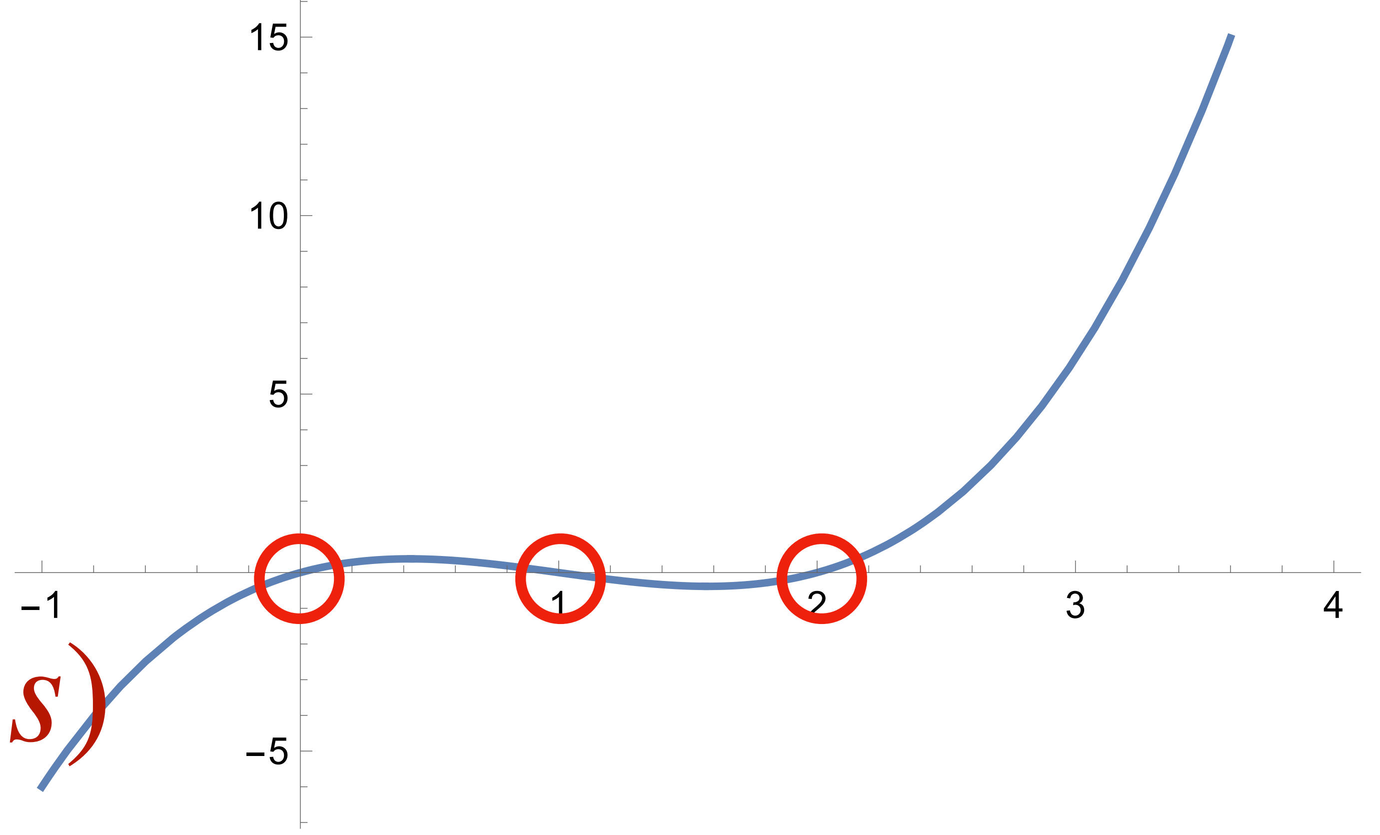

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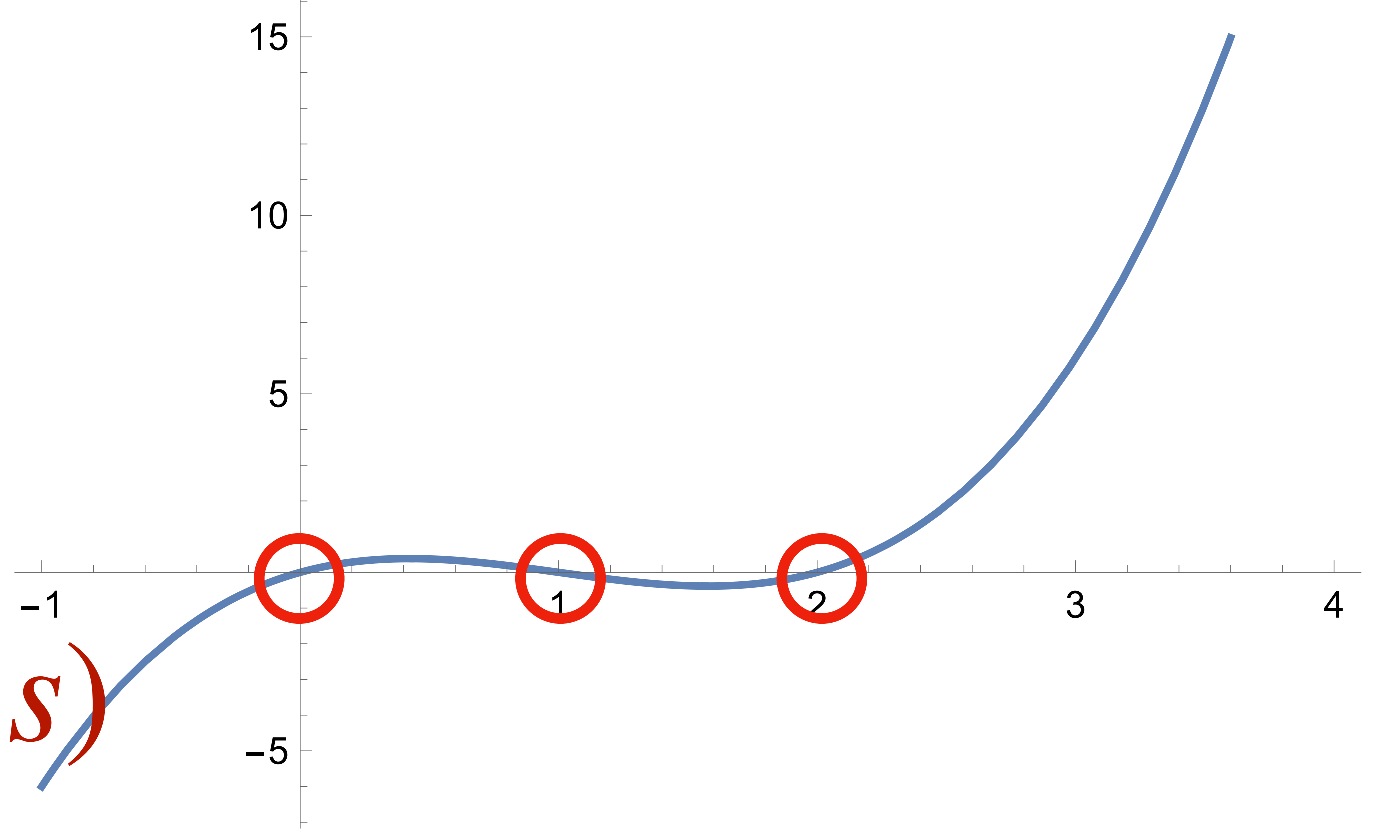

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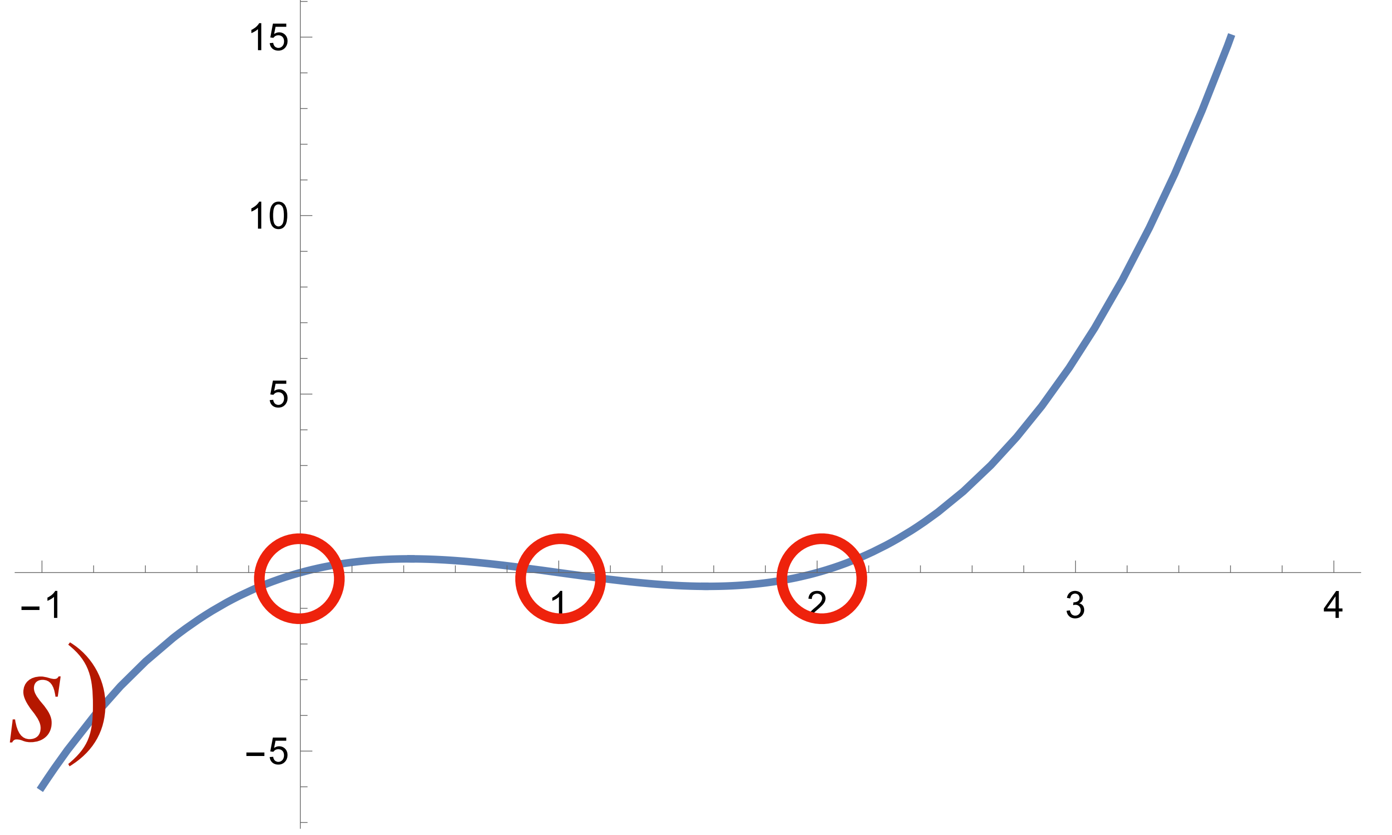

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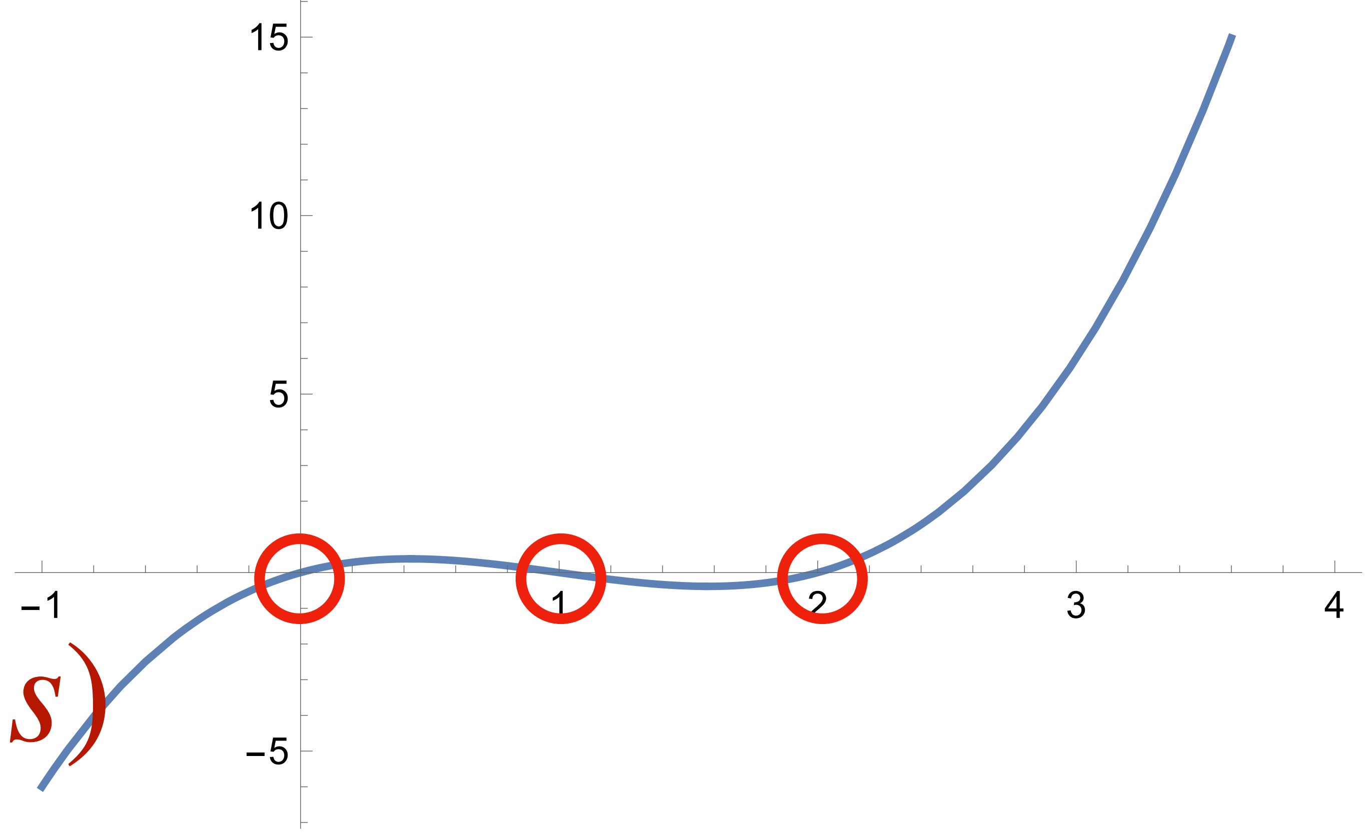

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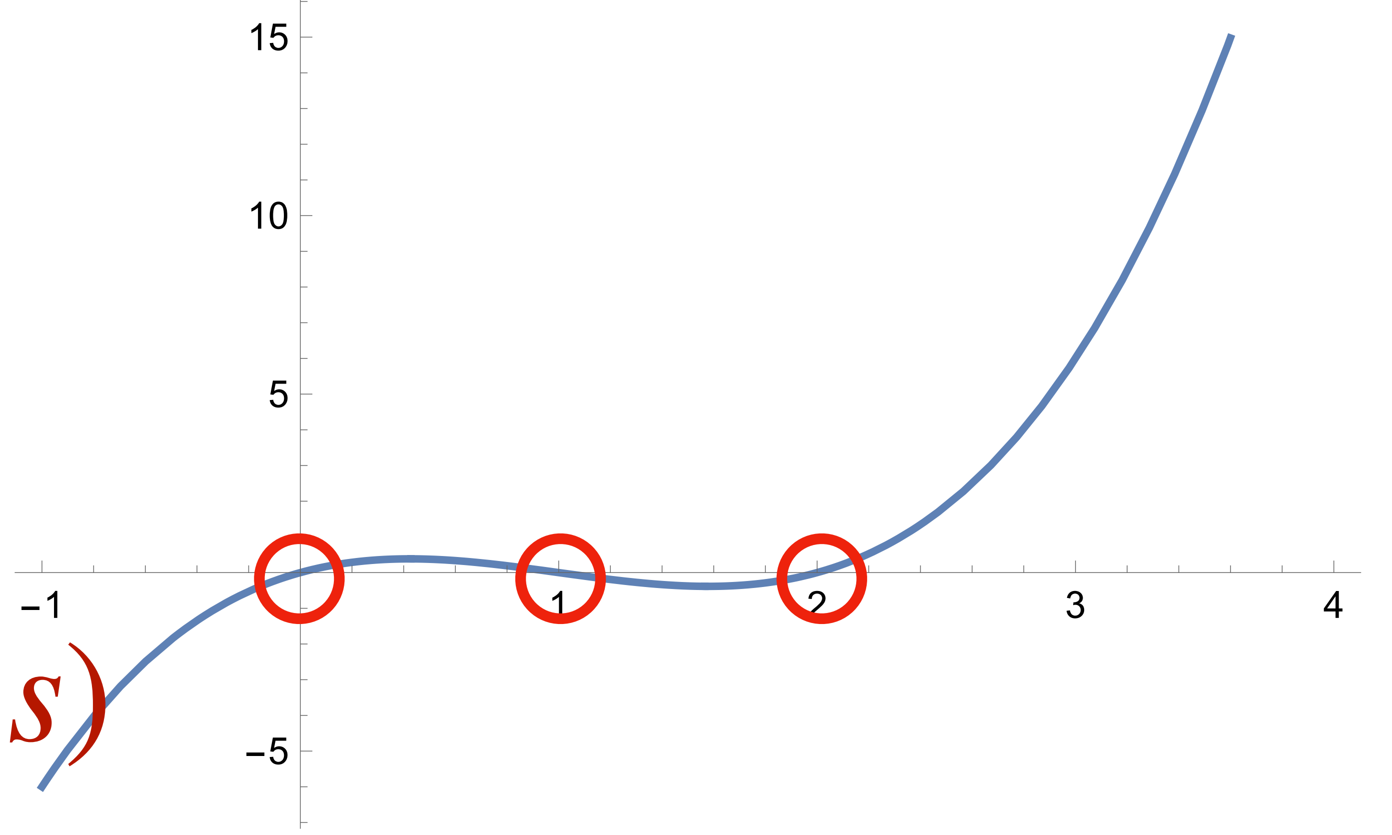

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  - Thus, $f(X) = q(X)(X - s)$ and $(X - s) \mid f(X)$
- **Lemma.** If $\deg(f) = N > n - 1$ and $f(X)$ vanishes on $\mathcal{S}$, then $f(X) = q(X)\mathbf{Z}_{\mathcal{S}}(X)$ for some $q(X) \in \mathbb{F}_{\leq N-n}[X]$

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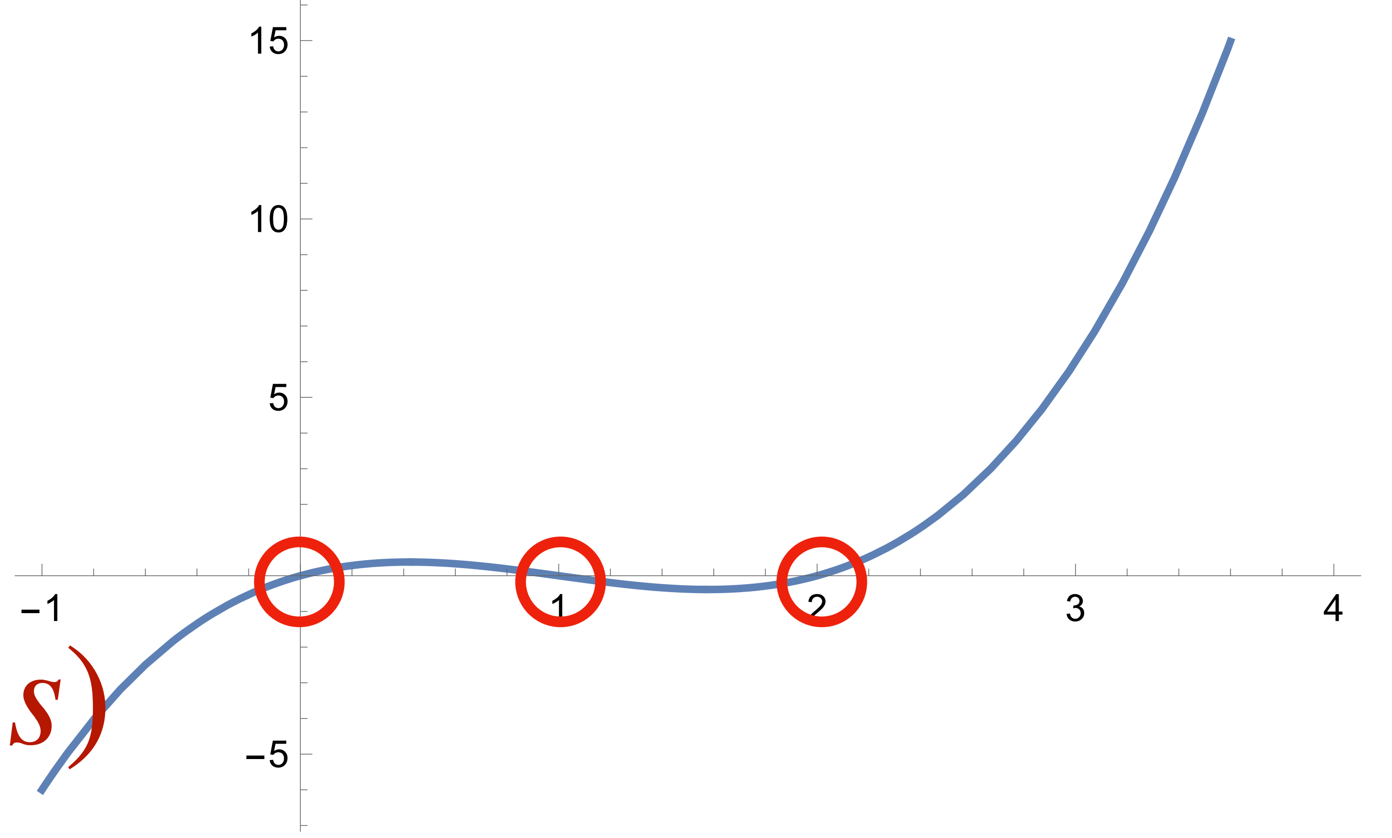

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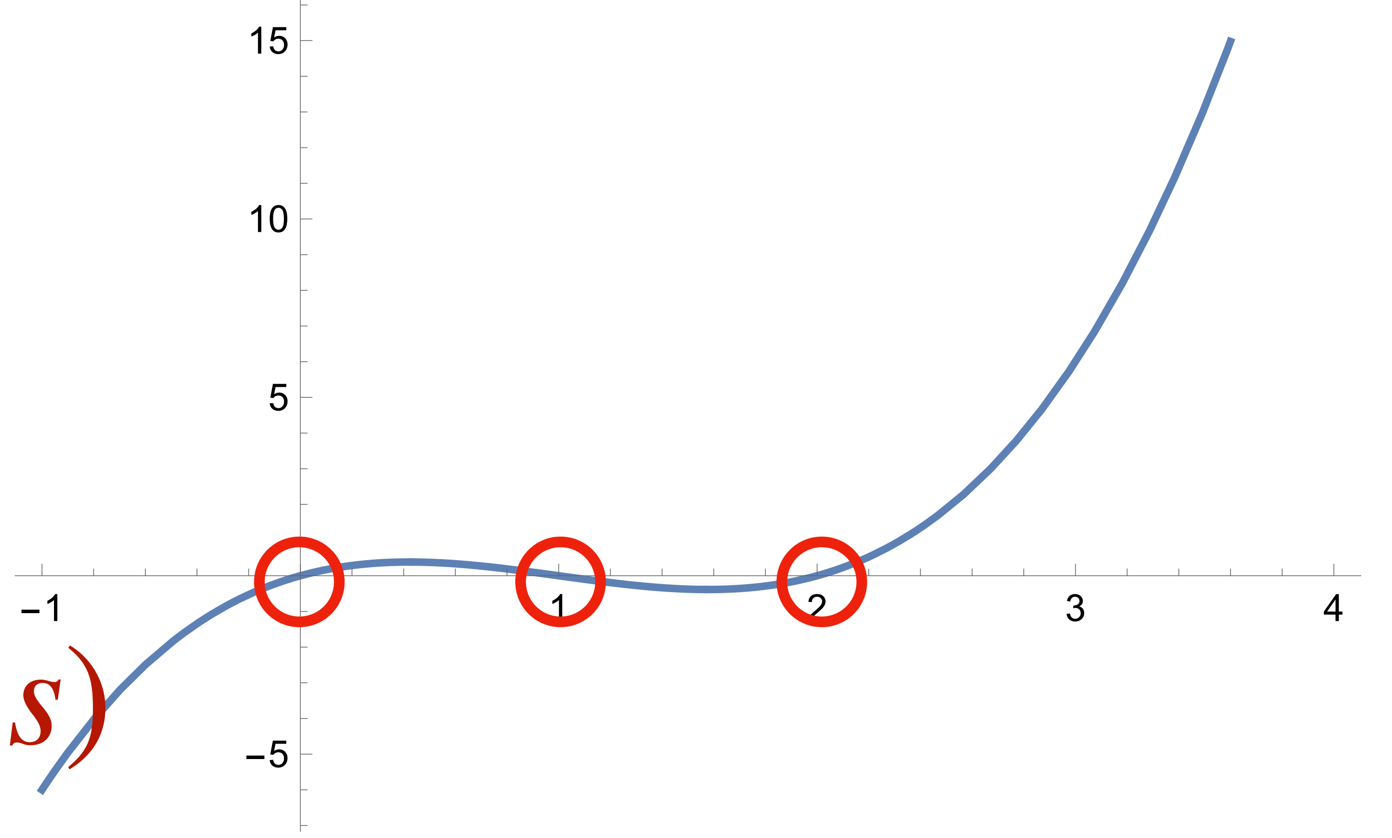
  - $\mathbf{Z}_{\mathbb{H}}(s) = 0$ for $s \in S$, $\mathbf{Z}_{\mathbb{H}}(s) \neq 0$ otherwise; $\deg(\mathbf{Z}_{\mathcal{S}}) = n$
- Any polynomial $f(X)$ that vanishes on $\mathcal{S}$ has all $s_i \in \mathcal{S}$ as roots
- Since $(X - s) \mid f(X)$ for all $s \in \mathcal{S}$ => $\mathbf{Z}_{\mathcal{S}}(X) \mid f(X)$
  - Assume $f(s) = 0$. Use polynomial long division (Extended Euclidean) to write $f(X) = q(X)(X - s) + r$ for a polynomial $q(X)$ and remainder $r \in \mathbb{F}$.
  - Evaluating LHS and RHS at $X = s$, we get $f(s) = r$, thus $r = 0$
  - Thus, $f(X) = q(X)(X - s)$ and $(X - s) \mid f(X)$
- **Lemma.** If $\deg(f) = N > n - 1$ and $f(X)$ vanishes on $\mathcal{S}$, then $f(X) = q(X)\mathbf{Z}_{\mathcal{S}}(X)$ for some $q(X) \in \mathbb{F}_{\leq N-n}[X]$
- $\mathbf{Z}_{\mathcal{S}}$ is **unique, minimal-degree, monic, non-zero** poly that vanishes exactly on $\mathcal{S}$

Vanishing Polynomials



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- $\mathbf{Z}_{\mathcal{S}}$ is **unique, minimal-degree, monic, non-zero** poly that vanishes exactly on $\mathcal{S}$
- **Important fact:** $\mathbf{Z}_{\mathbb{H}}(X) = \prod_i (X - \omega^{i-1}) = X^n - 1$ since $(\omega^{i-1})^n = \omega^{n(i-1)} = 1$

Polynomial View of Product Check

- $\forall i \in [1,n] \,.\, a_i b_i = c_i$

$\mathscr{R} = \{(\mathsf{x}, \mathsf{w}) : \mathsf{x} = [\![\hat{a}(X), \hat{b}(X), \hat{c}(X)]\!] \wedge \mathsf{w} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = (\mathsf{FFT}(\hat{a}(X)), \mathsf{FFT}(\hat{b}(X)), \mathsf{FFT}(\hat{c}(X))) \wedge \forall i \in [1,n] a_i b_i = c_i\}$

Polynomial View of Product Check

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Interpolation/polynomial evaluation: fast algorithms to get from witness to polynomial encoding and back

Polynomial View of Product Check

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  $\Leftrightarrow \exists q(X) \in \mathbb{F}_{\leq n-2}[X] \,.\, f(X) = q(X)\mathbf{Z}_{\mathbb{H}}(X)$ (1)

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- Prover needs to prove that it “knows” $\hat{a}(X), \hat{b}(X), \hat{c}(X)$ and $q(X)$ satisfying (1)

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Polynomial View of Product Check

Interpolation/polynomial evaluation: fast algorithms to get from witness to polynomial encoding and back

- NB! This is a standard way of using univariate polynomials —need to internalise it!

- $\forall i \in [1,n] \,.\, a_i b_i = c_i$

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Product Check PIOP

$\mathsf{x} = [\![\hat{a}(X), \bar{b}(X), \bar{c}(X)]\!], \mathsf{w} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{F}^n$

$\hat{a}(X) \leftarrow \mathsf{Interp}(\boldsymbol{a})$
$\hat{b}(X) \leftarrow \mathsf{Interp}(\boldsymbol{b})$
$\hat{c}(X) \leftarrow \mathsf{Interp}(\boldsymbol{c})$

$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$\leq n - 1$

$f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$
Virtual oracle:
$g(X) := f(X) - q(X)Z_{\mathbb{H}}(X)$

$\mathsf{x} = [\![\hat{a}(X)]\!], \hat{b}(X), \hat{c}(X)]\!]$

Product Check PIOP

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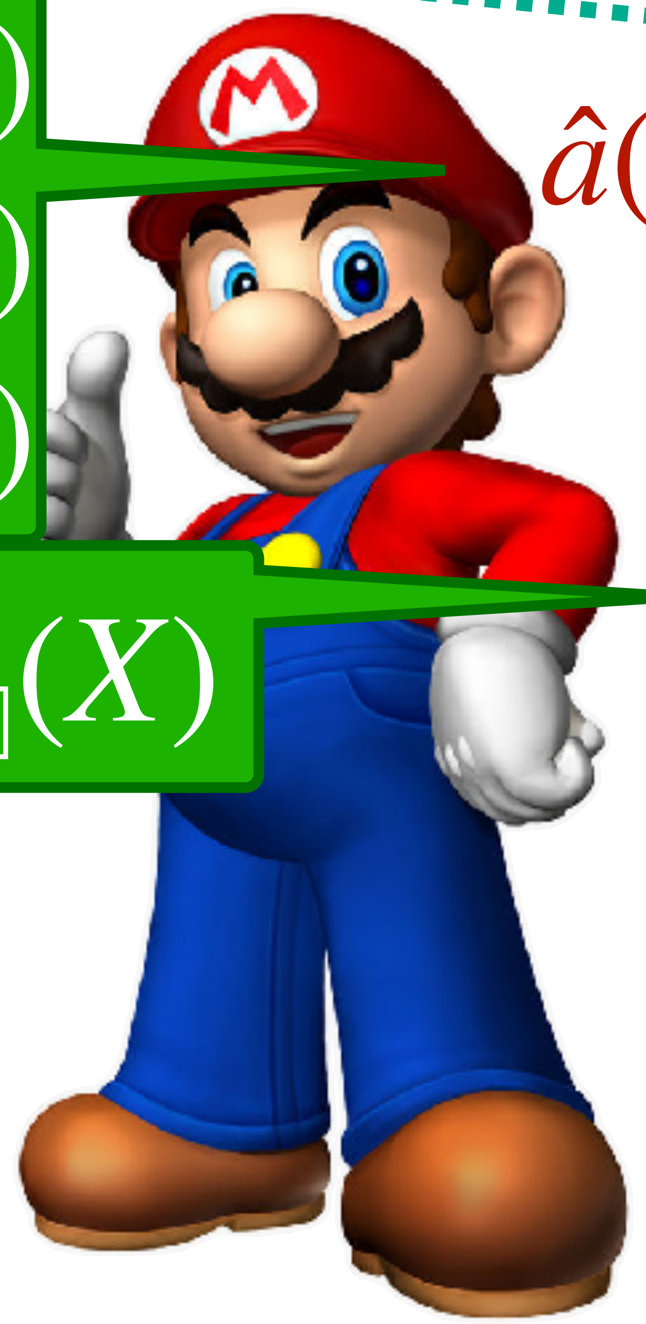
$\hat{c}(X) \leftarrow \mathsf{Interp}(\boldsymbol{c})$

$q(X) \leftarrow f(X)/\mathsf{Z}_{\mathbb{H}}(X)$



$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$q(X)$

$\leq n-1$

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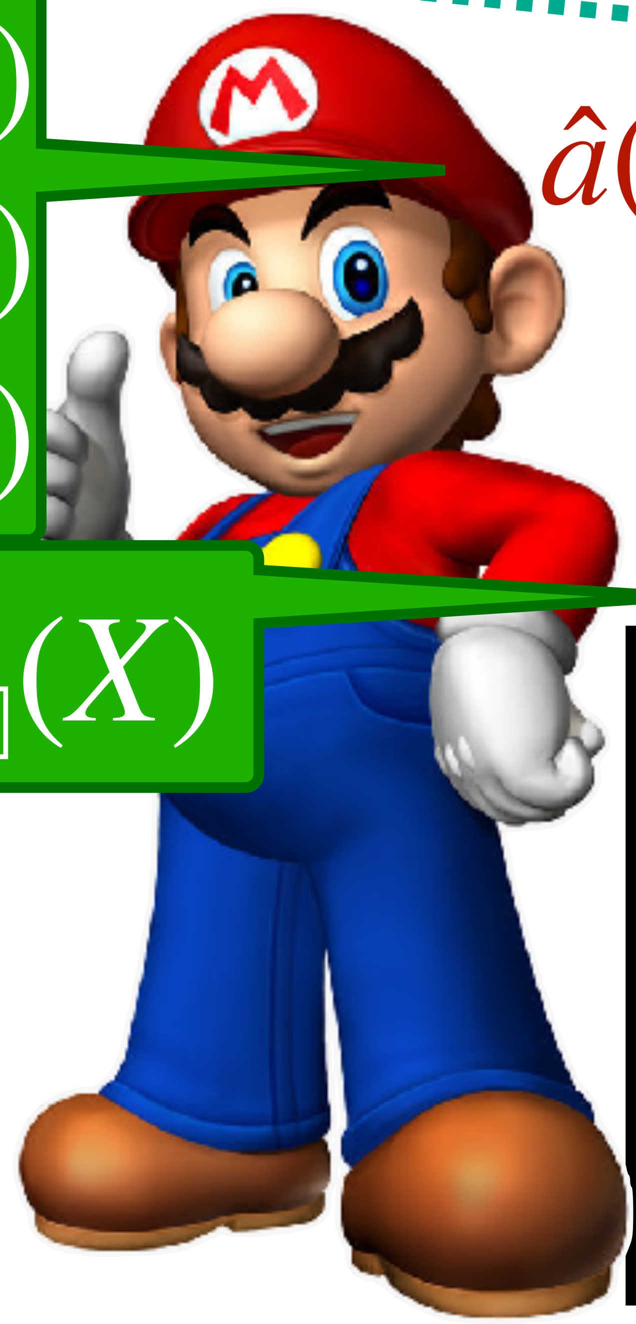

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$\hat{c}(X) \leftarrow \text{Interp}(\boldsymbol{c})$

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$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$q(X)$

$\leq n-1$

$\mathbb{x} = [\![\hat{a}(X)]\!], \hat{b}(X), \hat{c}(X)]\!]$

Zero Check for virtual oracle $[\![g(X)]\!]$

$r \leftarrow_{\$} \mathbb{F}$

$\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r)$

$\bar{c} \leftarrow \hat{c}(r), \bar{q} \leftarrow q(r)$

Product Check PIOP

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Virtual oracle:
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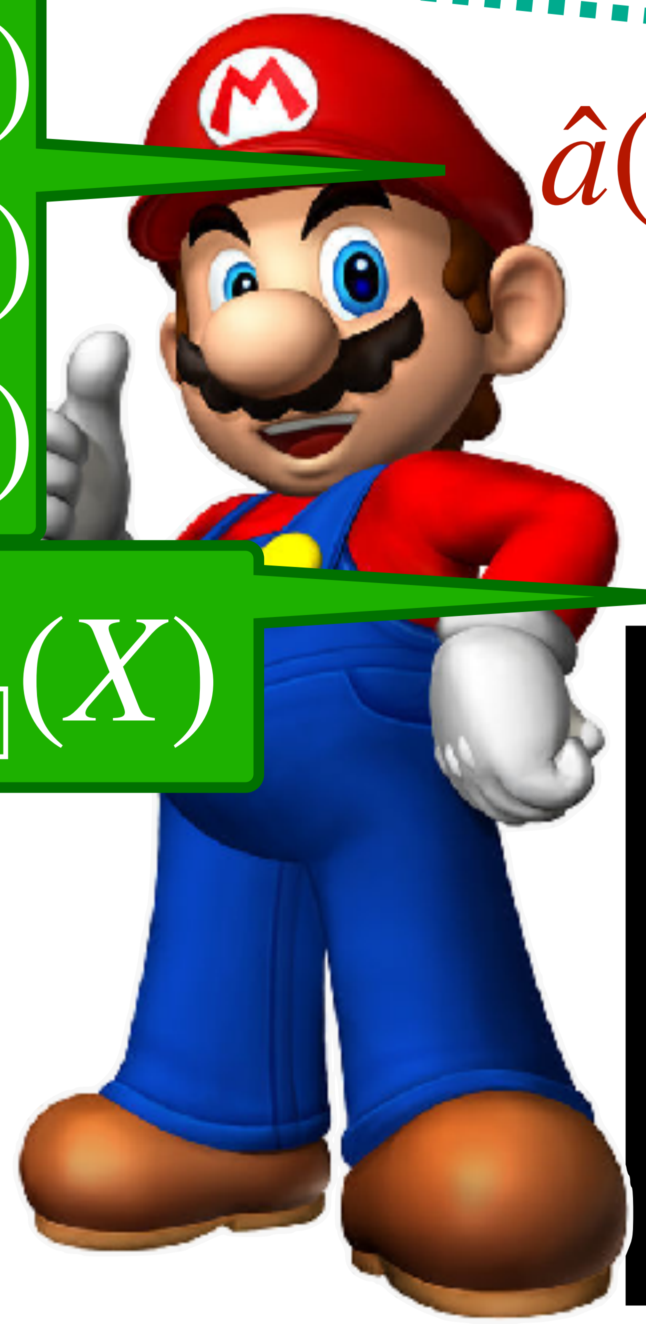

$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

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Zero Check for virtual oracle $[\![g(X)]\!]$

$r \leftarrow_\$ \mathbb{F}$

$\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r)$
$\bar{c} \leftarrow \hat{c}(r), \bar{q} \leftarrow q(r)$

Check $g(r) =^? 0$:
- $\mathbb{Z}_{\mathbb{H}}(r) \leftarrow r^n - 1$
- Check $\bar{a}\bar{b} - \bar{c} =^? \bar{q}\mathbb{Z}_{\mathbb{H}}(r)$

Product Check PIOP

$f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$

Virtual oracle:

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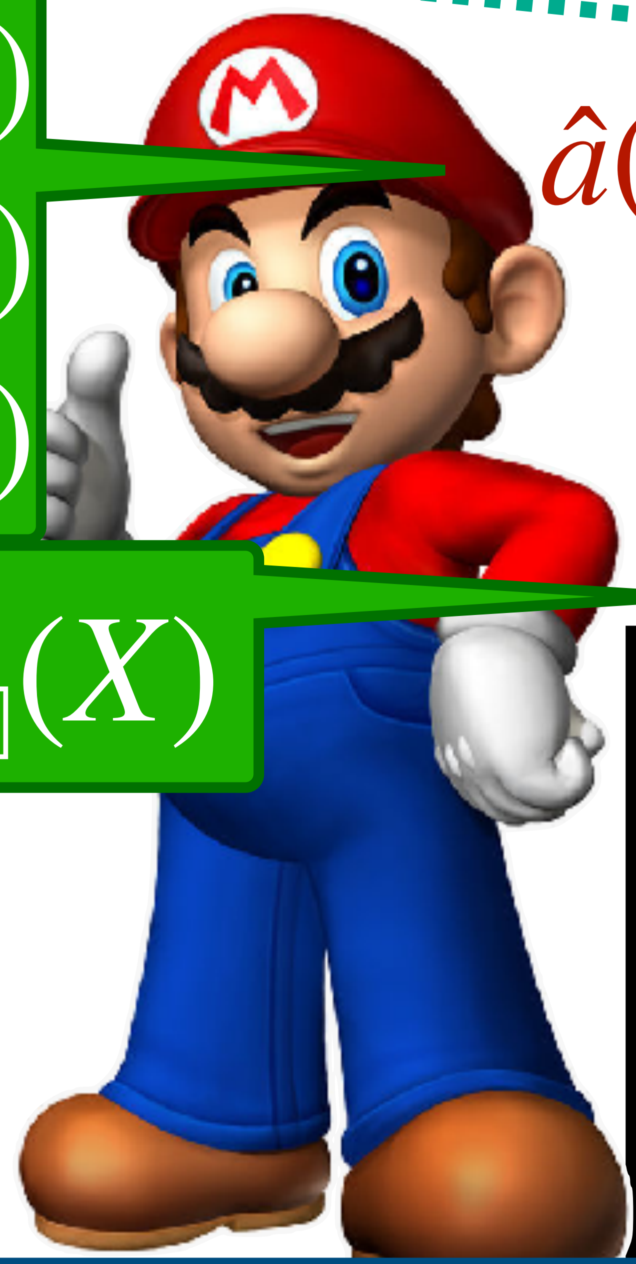

$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$q(X)$

$\leq n - 1$

$\mathrm{x} = [\![\hat{a}(X)], \hat{b}(X), \hat{c}(X)]\!]$

Zero Check for virtual oracle $[\![g(X)]\!]$

$r \leftarrow_{\$} \mathbb{F}$

$\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r)$

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Check $g(r) =^? 0$:

- $Z_{\mathbb{H}}(r) \leftarrow r^n - 1$
- Check $\bar{a}\bar{b} - \bar{c} =^? \bar{q}Z_{\mathbb{H}}(r)$

- $[\![g(X)]\!]$ is a <u>virtual</u> oracle
  - Not kept in an oracle but computed from other oracles
- V “virtually" queries $[\![g(X)]\!]$:
  - query $\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r), \bar{c} \leftarrow \hat{c}(r), \bar{q} \leftarrow q(r)$
  - set $g(r) \leftarrow \bar{a}\bar{b} - \bar{c} - \bar{q}Z_{\mathbb{H}}(r)$
  - check $g(r) =^? 0$

Efficiency

Product Check: Efficiency

$f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$
Virtual oracle:
$g(X) := f(X) - q(X)Z_{\mathbb{H}}(X)$

$\mathsf{x} = [\![\hat{a}(X), \bar{b}(X), \bar{c}(X)]\!], \mathsf{w} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{F}^n$

$\hat{a}(X) \leftarrow \text{Interp}(\boldsymbol{a})$
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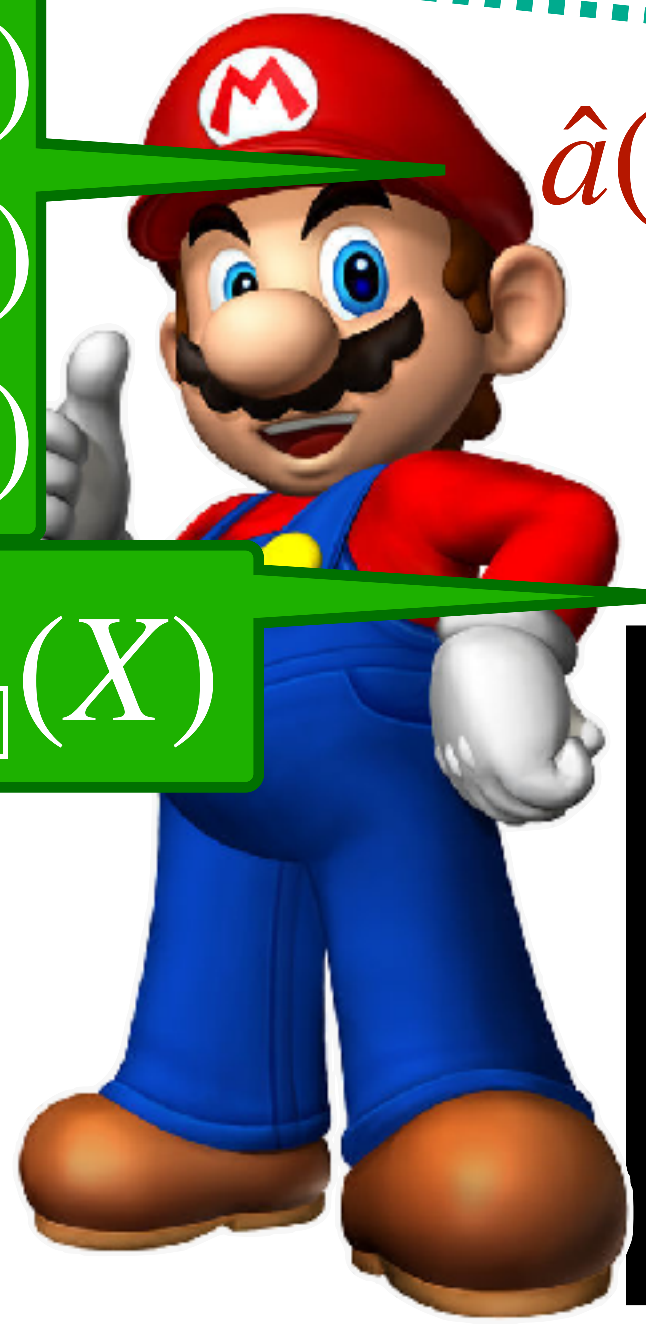

$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$\leq n - 1$

$q(X)$

$\mathsf{x} = [\![\hat{a}(X)]\!], \hat{b}(X), \hat{c}(X)]\!]$

Zero Check for virtual oracle $[\![g(X)]\!]$

$r \leftarrow_{\$} \mathbb{F}$

$\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r)$

$\bar{c} \leftarrow \hat{c}(r), \bar{q} \leftarrow$

Check $g(r) \stackrel{?}{=} 0$:
- $Z_{\mathbb{H}}(r) \leftarrow r^n - 1$
- Check $\bar{a}\bar{b} - \bar{c} \stackrel{?}{=} \bar{q}Z_{\mathbb{H}}(r)$

- One polynomial multiplication: $f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$
  - $O(n \log n)$ field ops // includes FFT & inverse FFT
- 3 interpolations: $\boldsymbol{a} \mapsto \hat{a}(X)$, …
  - Uses **inverse FFT**
- Total computation: $O(n \log n)$ field ops

Product Check: Efficiency

$f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$

Virtual oracle:

$g(X) := f(X) - q(X)Z_{\mathbb{H}}(X)$

$\mathsf{x} = [\![\hat{a}(X), \bar{b}(X), \bar{c}(X)]\!], \mathsf{w} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{F}^n$

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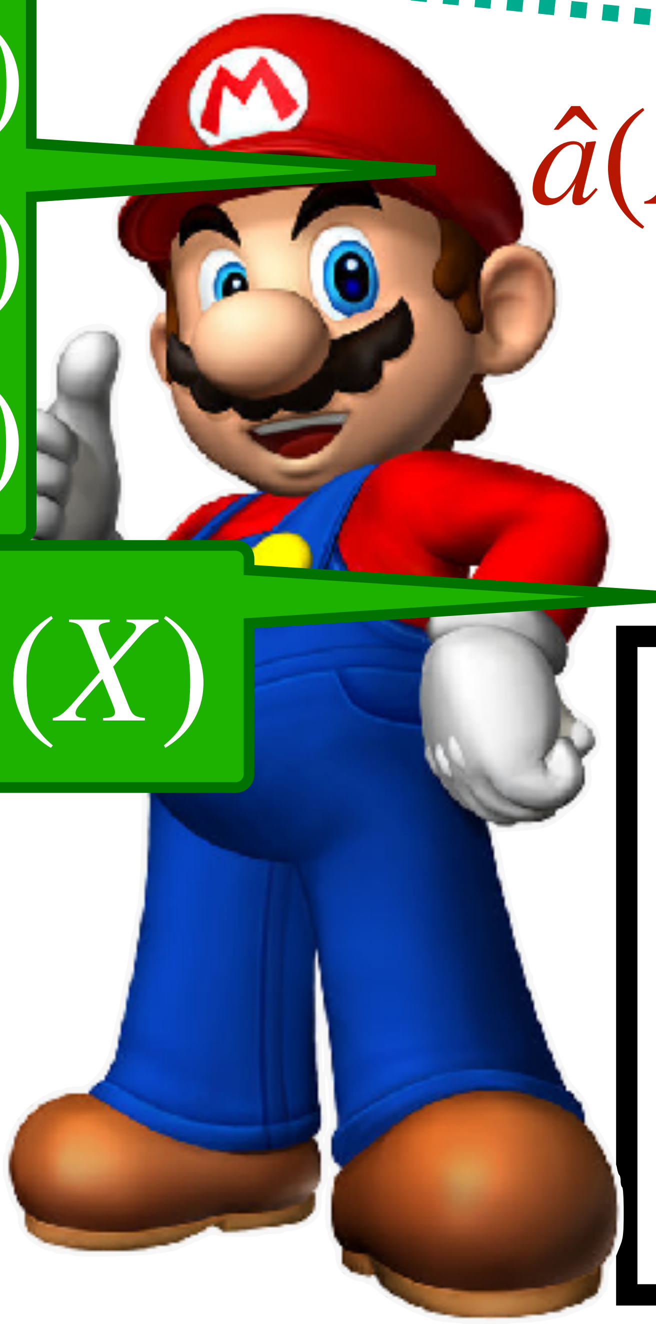

$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$\leq n-1$

$\mathsf{x} = [\![\hat{a}(X)], \hat{b}(X), \hat{c}(X)]\!]$

$q(X)$

Zero Check for virtual oracle $[\![g(X)]\!]$

$r \leftarrow_{\$} \mathbb{F}$

$\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r)$

$\bar{c} \leftarrow \hat{c}(r), \bar{q}$

Check $g(r) =^? 0$:

- $Z_{\mathbb{H}}(r) \leftarrow r^n - 1$
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- One polynomial multiplication: $f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$
  - $O(n \log n)$ field ops // includes FFT & inverse FFT
- 3 interpolations: $\boldsymbol{a} \mapsto \hat{a}(X), \ldots$
  - Uses **inverse FFT**
- Total computation: $O(n \log n)$ field ops

- Computing $Z_{\mathbb{H}}(r)$ // $O(\log n)$ f.o.
- +2 multiplications
- Total time $O(\log n)$ field ops

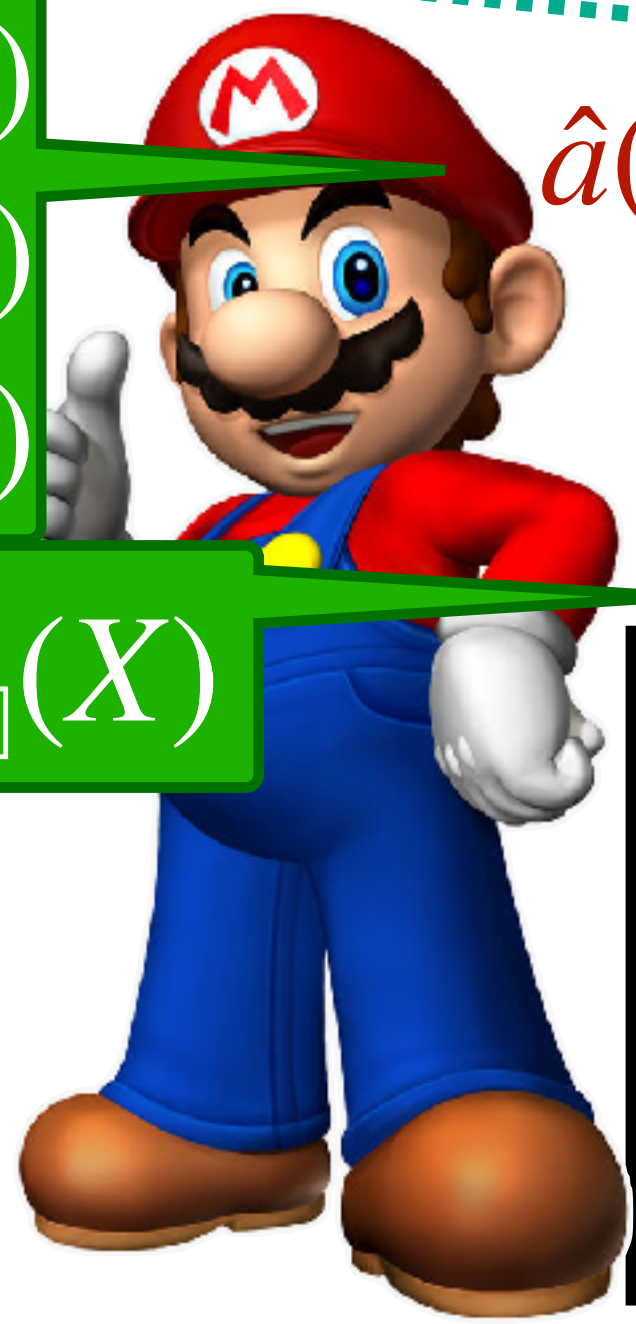
Product Check: Efficiency

$f(X) := \hat{a}(X)\hat{b}(X) - \hat{c}(X)$
Virtual oracle:
$g(X) := f(X) - q(X)Z_{\mathbb{H}}(X)$

$\mathbb{x} = [\![\hat{a}(X), \bar{b}(X), \bar{c}(X)]\!], \mathbb{w} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{F}^n$

$\hat{a}(X) \leftarrow \text{Interp}(\boldsymbol{a})$
$\hat{b}(X) \leftarrow \text{Interp}(\boldsymbol{b})$
$\hat{c}(X) \leftarrow \text{Interp}(\boldsymbol{c})$

$q(X) \leftarrow f(X)/Z_{\mathbb{H}}(X)$



$\hat{a}(X), \hat{b}(X), \hat{c}(X)$

$\leq n - 1$

$q(X)$

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Zero Check for virtual oracle $[\![g(X)]\!]$

$r \leftarrow_{\$} \mathbb{F}$

$\bar{a} \leftarrow \hat{a}(r), \bar{b} \leftarrow \hat{b}(r)$
$\bar{c} \leftarrow \hat{c}(r), \bar{q} \leftarrow q(r)$

Check $g(r) \stackrel{?}{=} 0$:
- $Z_{\mathbb{H}}(r) \leftarrow r^n - 1$
- Check $\bar{a}\bar{b} - \bar{c} \stackrel{?}{=} \bar{q}Z_{\mathbb{H}}(r)$

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- **Memory locality:** butterfly-like memory access causes cache misses

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- **Trade-off:** in “non-P“IOPs, one can instantiate crypto more efficiently

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- One more seminar: Fiat-Shamir (how to make it non-interactive)

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- Code&hash-based => using any fields, hash is fast, post-quantum

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- Formal verification and automated security proofs

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Questions?

Important References

- (FFT) James W. Cooley, John W. Tukey: **An algorithm for the machine calculation of complex Fourier series** (1965)
    - Classic algorithm, many brilliant presentations, including on YouTube
- (Good book on polynomial algorithms) Joachim von zur Gathen, Jürgen Gerhard: **Modern Computer Algebra (3. ed.).** Cambridge University Press 2013
- PIOP:
    - Benedikt Bünz, Ben Fisch, Alan Szepieniec. **Transparent SNARKs from DARK compilers** (2020)
    - Alessandro Chiesa, Yuncong Hu, Mary Maller, Pratyush Mishra, Noah Vesely, Nicholas Ward. **Marlin: Preprocessing zkSNARKs with Universal and Updatable SRS** (2020)