Query Evaluation: Basics and Recent Developments

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Short bio:

- PhD (2015) from the University of Helsinki in mathematical logic
- Research during PhD/postdoc:
  - Dependence logic $\forall, \exists, \wedge, \vee, \neg, =\!(x, y)$
  - Implication problem $\Sigma \models \tau$? for database dependencies
- Since 2024 assoc. prof. in data management at Tartu

This talk: General overview[1] of one of the most fundamental problems in database theory:

- Query evaluation

[1] Main source: [Arenas et al., 2022]

Introduction

Complexity of CQ-Evaluation

Joins with Information Theory

Conclusion

Output of a query

```
SELECT e.emp_id, e.emp_name
FROM Employees AS e, Departments AS d
WHERE e.dep_id = d.dep_id AND d.dep_name = 'Sales';
```

Employees

emp_id	emp_name	dep_id
12345	Alice	10
67890	Bob	20
23456	Charlie	10

Departments

dep_id	dep_name
10	Sales
20	Engineering

$\longrightarrow$

Output

emp_id	emp_name
12345	Alice
23456	Charlie

Notation $q(D)$ = output of query q on database D

Problem:	Query-Evaluation
Input:	A query q, a database D, a tuple of values $\bar{a}$
Output:	`true` if $\bar{a} \in q(D)$, and `false` otherwise

Previous example:

Input:

- query q = given SQL query
- database D = given database
- tuple $\bar{a}$ = (Alice, Sales)

Output: `true` / (12345, Alice)$\in q(D)$

Notation $q(D)$ = output of query q on database D

Problem:	Query-Evaluation
Input:	A query q, a database D, a tuple of values $\bar{a}$
Output:	`true` if $\bar{a} \in q(D)$, and `false` otherwise

Previous example:

Input:

- query q = given SQL query
- database D = given database
- tuple $\bar{a}$ = (Alice, Sales)

Output: `true` / (12345, Alice)$\in q(D)$

Mathematics behind

How to analyse the complexity of Query-Evaluation?

We need a mathematical description of a

- (relational) database
- (SQL) query
- (data tuple)

Database

Employees

emp_id	emp_name	dep_id
12345	Alice	10
67890	Bob	20
23456	Charlie	10

Departments

dep_id	dep_name
10	Sales
20	Engineering

- Each entry viewed as a fact, i.e., an expression such as Employees(12345, Alice, 10)
  - let's use here a shorthand: Emp(12345, Alice, 10)
- A database D defined as a finite set of facts:

  {Emp(12345, Alice, 10), Emp(67890, Bob, 20), Emp(23456, Charlie, 10), Dep(10, Sales), Dep(20, Engineering)}

- D is a database of a schema $\mathbf{S} = \{E[3], D[2]\}$ specifying its structure

Query

A query q over schema **S** is a function that maps databases D of **S** to finite sets of sequences (of the same length)

$$q(D) = \{(a_1, \ldots, a_k), (b_1, \ldots, b_k), \ldots\}$$

Example

In our example, $q(D) = \{(12345, \text{Alice}), (23456, \text{Charlie})\}$

Such queries can be described using query languages. Two paradigms:

- Declarative languages (logic)
- Procedural languages (algebra)

Queries and logic

To simplify analysis, let us restrict attention to the so-called "Core SQL", formed using only commands `SELECT`, `FROM`, `WHERE` with equality comparisons

vs.

First-order logic (FO):

- Atomic formulas $R(x, y), x = y, \ldots$
- Connectives $\wedge, \vee, \neg$
- Quantifiers $\exists, \forall$

Queries and logic

To simplify analysis, let us restrict attention to the so-called "Core SQL", formed using only commands `SELECT`, `FROM`, `WHERE` with equality comparisons

$=$

Conjunctive query (CQ):

- Atomic formulas $R(x, y)$, ~~$x = y, \ldots$~~
- Connectives $\wedge$, ~~$\vee, \neg$~~
- Quantifiers $\exists$, ~~$\forall$~~

Queries and logic

```
SELECT e.emp_id, e.emp_name
FROM Employees AS e, Departments AS d
WHERE e.dep_id = d.dep_id AND d.dep_name = 'Sales';
```

. . . corresponds to the CQ . . .

$$\phi(x, y) := \exists z \exists w(\mathrm{Emp}(x, y, z) \wedge \mathrm{Dep}(z, \text{'Sales'}))$$

where

- Variables x, y are free and correspond to the output
- Variables z is bound by $\exists$; 'Sales' called a constant

Queries and logic

```
SELECT e.emp_id, e.emp_name
FROM Employees AS e, Departments AS d
WHERE e.dep_id = d.dep_id AND d.dep_name = 'Sales';
```

. . . corresponds to the CQ . . .

$$\text{Answer}(x, y) :- \text{Emp}(x, y, z), \text{Dep}(z, \text{'Sales'})$$

where

- $\text{Emp}(x, y, z)$, $\text{Dep}(z, \text{'Sales'})$ forms the body
- $\text{Answer}(x, y)$ forms the head

CQ Semantics

Given a database

$$D = \{\text{Emp}(12345,\ \text{Alice},\ 10),\ \text{Emp}(67890,\ \text{Bob},\ 20),\ \text{Emp}(23456,\ \text{Charlie},\ 10),\ \text{Dep}(10,\ \text{Sales}),\ \text{Dep}(20,\ \text{Engineering})\}$$

and a query

$$q = \text{Answer}(x, y) \coloneq \text{Emp}(x, y, z), \text{Dep}(z, \text{'Sales'})$$

we define the output $q(D)$ as the set of all pairs $(h(x), h(y))$, where

- h is a mapping from variables to constants, and
- $\text{Emp}(h(x), h(y), h(z))$ and $\text{Dep}(h(z), \text{'Sales'})$ are in D

This can be represented...

```
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$\longrightarrow$

Output

emp_id	emp_name
12345	Alice
23456	Charlie

...as this

- query $q = \text{Answer}(x, y) \coloneq \text{Emp}(x, y, z), \text{Dep}(z, \text{'Sales'})$
- database
  $D = \{$Emp(12345, Alice, 10), Emp(67890, Bob, 20), Emp(23456, Charlie, 10), Dep(10, Sales), Dep(20, Engineering)$\}$
- output $q(D) = \{(12345, \text{Alice}), (23456, \text{Charlie})\}$

Query-Evaluation revisited

Problem:	Query-Evaluation
Input:	A query q, a database D and a tuple of values $\bar{a}$
Output:	`true` if $\bar{a} \in q(D)$, and `false` otherwise

Our example:

Input:

- query $q = \text{Answer}(x, y) \coloneq \text{Emp}(x, y, z), \text{Dep}(z, \text{'Sales'})$
- database
  $D = \{$Emp(12345, Alice, 10), Emp(67890, Bob, 20), Emp(23456, Charlie, 10), Dep(10, Sales), Dep(20, Engineering)$\}$
- tuple $\bar{a} = (12345, \text{Alice})$

Output: `true`

CQ-Evaluation

Problem:	CQ-Evaluation
Input:	A Boolean conjunctive query q, a database D
Output:	`true` if D satisfies q, and `false` otherwise

Our example:

Input:

- query $q = \text{Answer} \coloneq \text{Emp}(12345, \text{'Alice'}, z), \text{Dep}(z, \text{'Sales'})$
- database
  $D = \{\text{Emp}(12345,\ \text{Alice},\ 10),\ \text{Emp}(67890,\ \text{Bob},\ 20),\ \text{Emp}(23456,\ \text{Charlie},\ 10),$
  $\text{Dep}(10,\ \text{Sales}),\ \text{Dep}(20,\ \text{Engineering})\}$

Output: `true`

Introduction

Complexity of CQ-Evaluation

Joins with Information Theory

Conclusion

Theorem

CQ-Evaluation is NP*-complete.*

Proof.

CQ-Evaluation is in NP: Guess the function h from variables to constants and check that the body of q is mapped into D.

CQ-Evaluation is in NP-hard: Next page. □

Hardness: Reduction from Clique to CQ-Evaluation

Clique:

- Given a natural number k and an undirected graph G with vertex set V and edge set E (without self-loops $\{u, u\}$), decide if G has a clique of size k.
- NP-complete

CQ-Evaluation:

- Construct a database D as follows:

$$D = \{\text{Node}(v) \mid v \in V\} \cup \{\text{Edge}(u, v) \mid \{u, v\} \in E\}$$

$$q = \exists x_1 \ldots \exists x_k \left(\bigwedge_{i=1}^{k} \text{Node}(x_i) \wedge \bigwedge_{\substack{i,j \in [k] \\ i \neq j}} \text{Edge}(x_i, x_j) \right)$$

$\Longrightarrow$ D satisfies q if and only if G contains a clique of size k

Hardness analysed

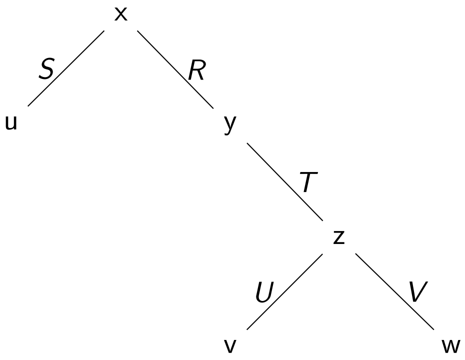
- Hardness of CQ-Evaluation can arise from queries shaped as **cliques**
- Such queries not common / typically queries shaped as **trees**

Hardness analysed cont.

Consider

$$\text{Answer} \coloneq S(u,x), R(x,y), T(y,z), U(z,v), V(z,w)$$

having shape:



Semi-Join

For two relations R and S the semi-join $R \ltimes S$ returns all rows from R that have matching rows in S

Example

R	x	y
	1	a
	2	b
	3	c
	4	d

$\ltimes$

T	y	z
	a	10
	a	10
	c	20
	e	30

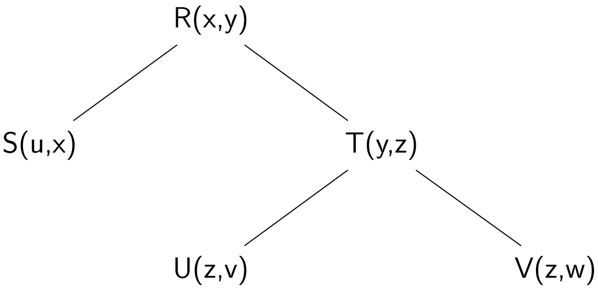
$=$

	x	y
	1	a
	3	c

Evaluating tree queries

Computation of Answer:

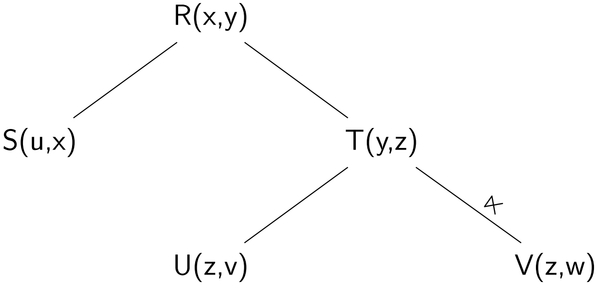
1. Take the line graph of prev. graph



Evaluating tree queries

Computation of Answer:

1. Take the line graph of prev. graph
2. $T[y,z] := T[y,z] \ltimes V[z,w]$



Evaluating tree queries

Computation of Answer:

1. Take the line graph of prev. graph
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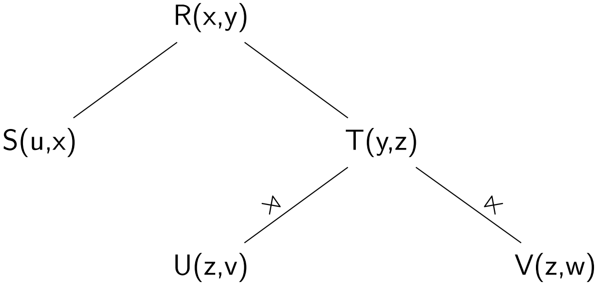

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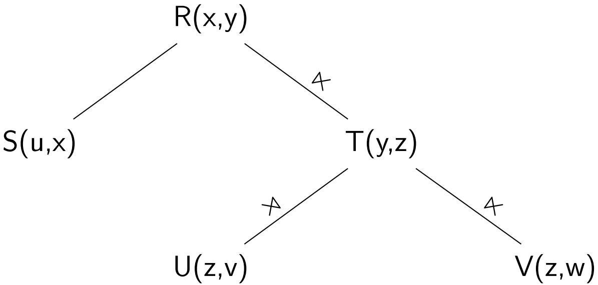

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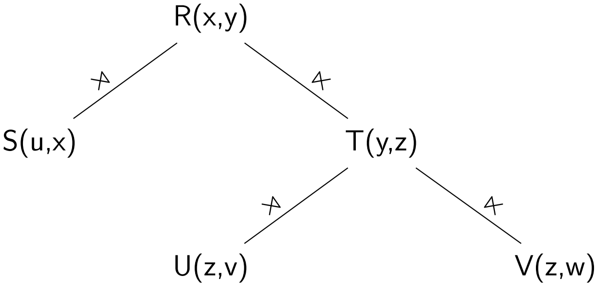

Evaluating tree queries

Computation of Answer:

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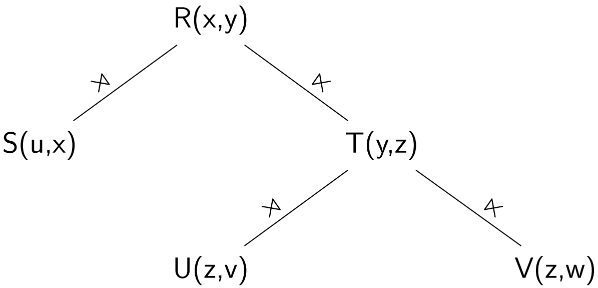
$\rightarrow$ Answer is `true` iff R is non-empty in the end

Time complexity:
$O(||D|| \cdot \log ||D|| \cdot ||q||)$



Yannakakis

- Previous algorithm known as the Yannakakis algorithm [Yannakakis, 1981]
- Follows a bottom-up dynamic programming approach

But, we can do better (Yannakakis algorithm is actually more general, as we will see):

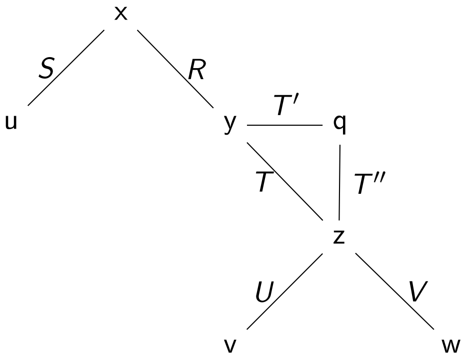
- What if q contains only **small** cliques?

Small clique

Consider

$$\text{Answer} :- S(u,x), R(x,y), T(y,z), T'(y,q), T''(q,z), U(z,v), V(z,w)$$

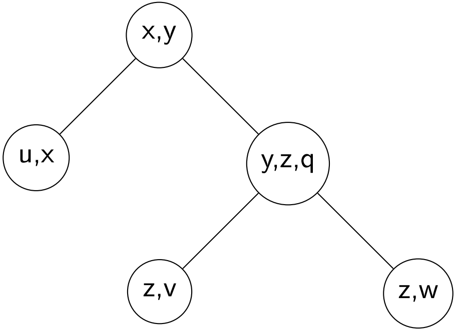
having shape:



Small clique re-organised

Atoms are grouped into nodes:

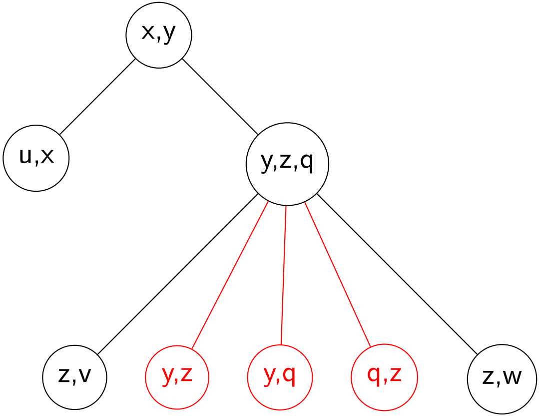
- $S(u, x)$
- $R(x, y)$
- $T(y, z)$, $T'(y, q)$, $T''(q, z)$
- $U(z, v)$
- $V(z, w)$



Small clique further re-organised

Add 3 new children for the clique
$\rightarrow$

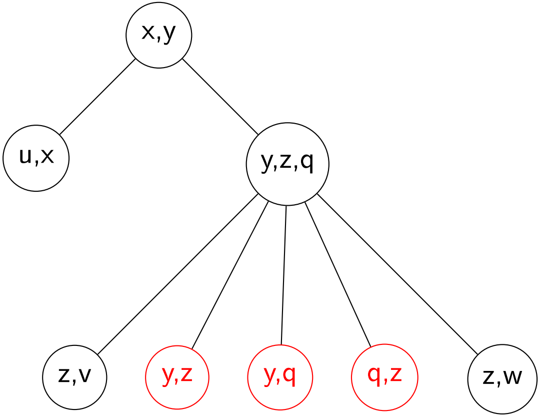
- $S(u, x)$
- $R(x, y)$
- $T(y, z)$, $T'(y, q)$, $T''(q, z)$
- $U(z, v)$
- $V(z, w)$



Computation of answer

adom(D) = the set of all values appearing in D

1. $A[y, z, q] := \text{adom}(D)^3$



Computation of answer

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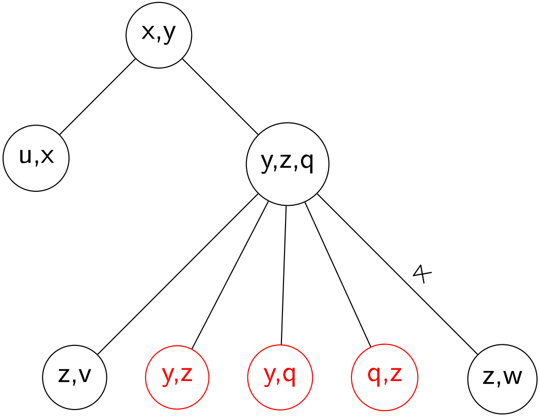

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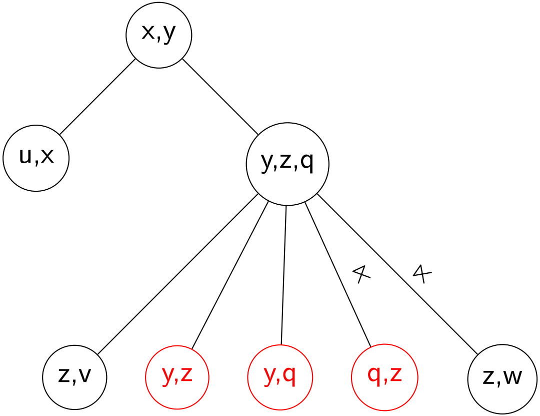

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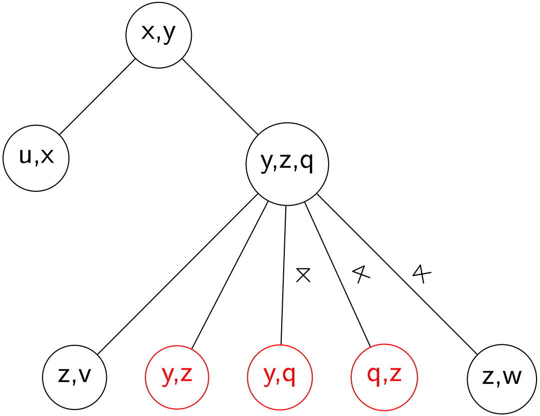

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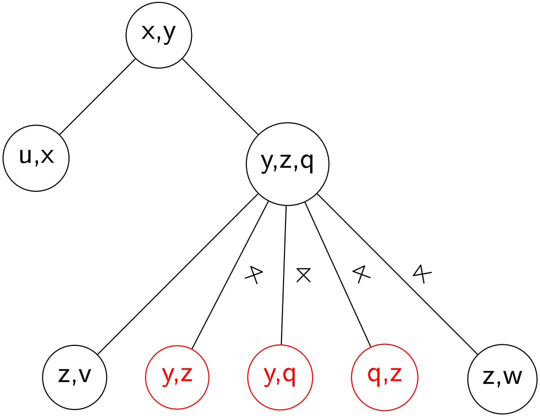

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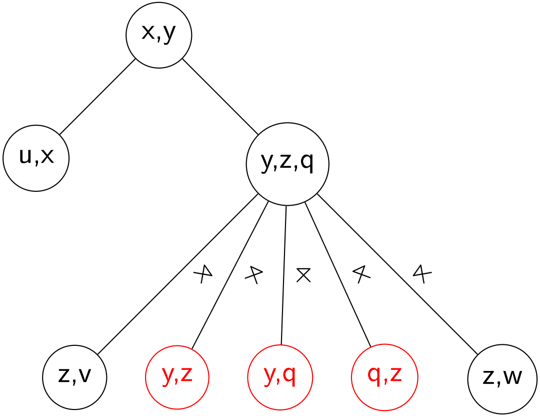

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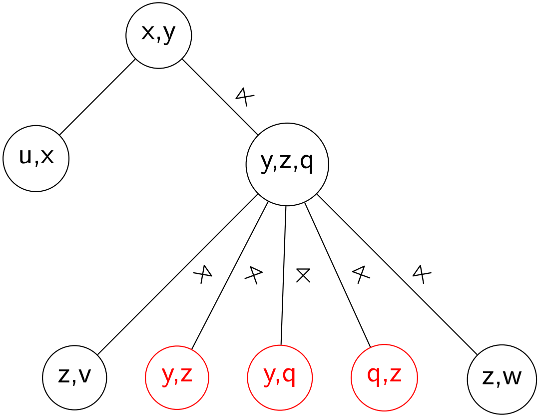

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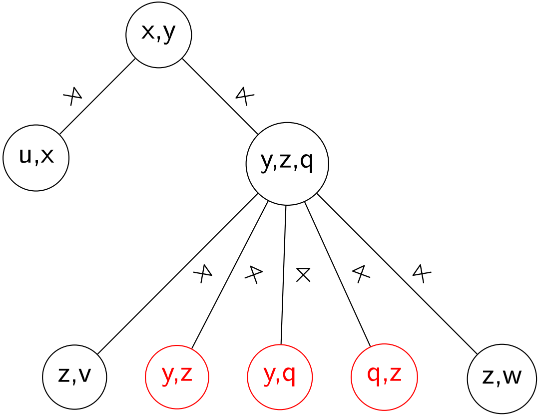

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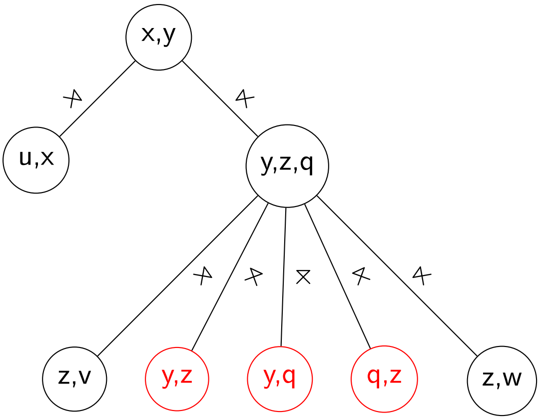
8. $R[x, y] := R[x, y] \ltimes S[u, x]$

Answer = `true` iff R non-empty in the end. Time complexity (here) $O((||q|| + 1)(||D||^3 \log ||D||^3))$



CQ as a hypergraph

Let's generalise this idea:

- Queries can have more than two terms in an atom $\rightarrow$ hypergraphs
- Hypergraphs can be re-structured as a tree that group $\leq k$ variables (leading to the $||D||^k$-factor in time complexity) $\rightarrow$ treewidth to measure growth of complexity

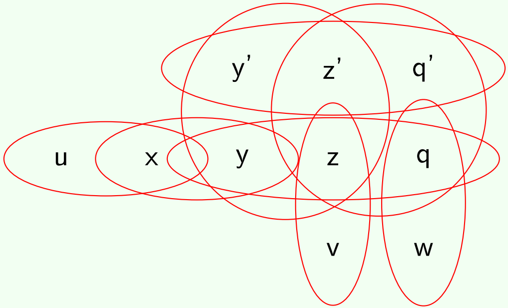
Definition

A hypergraph is a pair $H = (V, E)$ consisting of a set V of nodes and a set E of subsets of V, called hyperedges.

Example

$$\text{Answer} :- S(u,x), R(x,y), T(y,z,q), U(z,v), V(z,w),$$
$$A(y',z',q'), B(y,z,y',z'), C(z,q,z',q')$$

as a hypergraph:



Treewidth

Definition

A tree decomposition of a hypergraph $G = (V, E)$ is a tree $T = (B, E_T)$, where the vertex set $B \subseteq \mathcal{P}(V)$ is a collection of bags and E_T is a set of edges as follows:

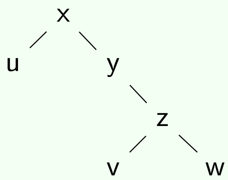
1. $\bigcup_{b \in B} b = V$,
2. for every $e \in E$ there is a bag $b \in B$ with $e \subseteq b$, and
3. for all $v \in V$ the subtree of T induced by the bags containing v is connected.

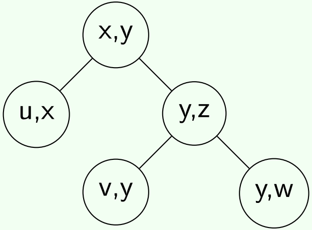
The width of the tree decomposition T is the size of the largest bag decreased by one: $\max_{b \in B} |b| - 1$. The treewidth of G is the minimum width over all tree decompositions of G.

G is tree iff it has treewidth 1

Example

Left: our first "tree query". Right: its tree decomposition of width 1.

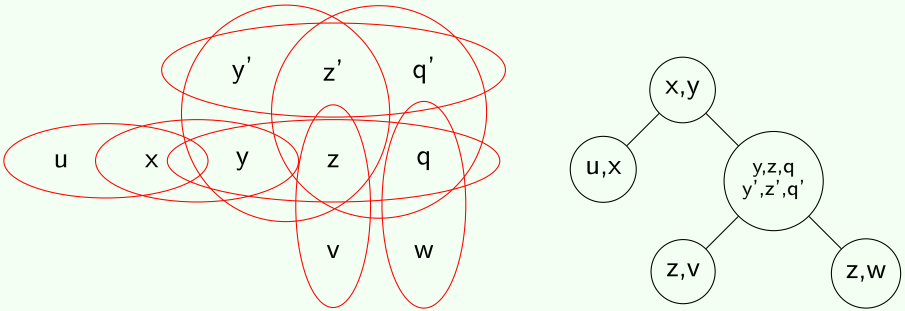




Last example

Example

Left: our last example query. Right: its tree decomposition of width 5.



CQ-Evaluation for fixed treewidth

Theorem

Fix $k \geq 1$. Then CQ-Evaluation, restricted to queries with treewidth at most k, can be solved in time

$$O(||D||^{k+1} \cdot ||q||^4 \cdot (\log ||D|| + \log ||q||)).$$

Key pointers:

- Tree decomposition can be constructed in time $2^{O(k^3)}||G||$ for a (hyper)graph G with treewidth k [Bodlaender, 1996] $\implies$ linear time for fixed k
- Treewidth = largest bag size - 1 $\implies$ $||D||^{k+1}$-factor

Happy?

Not happy. Efficient evaluation can be possible even with unbounded treewidth

Example

Consider "n-clique"+"one giant atom":

$$\text{Answer} :- R(x_i, x_j)_{\substack{i,j\in[n]\\ i\neq j}}, S(x_1, \ldots, x_n)$$

Treewidth is $n-1$, yet the following procedure is efficient:

- Step 1. $S[x_1, \ldots, x_n] := S[x_1, \ldots, x_n] \ltimes R[x_1, x_1]$
- Step 2. $S[x_1, \ldots, x_n] := S[x_1, \ldots, x_n] \ltimes R[x_1, x_2]$
- ...
- Step n^2. $S[x_1, \ldots, x_n] := S[x_1, \ldots, x_n] \ltimes R[x_n, x_n]$

Yannakakis revisited

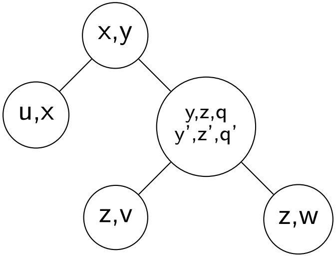
The max size of a bag (in a tree decomposition) is not crucial; the number of hyperedges covering a bag is. Bottom right: tree decomposition of our example query

$$\text{Answer} :- S(u,x), R(x,y), T(y,z,q), U(z,v), V(z,w),$$
$$A(y',z',q'), B(y,z,y',z'), C(z,q,z',q')$$

Computation of Answer

1. $X[y,z,q,y',z',q'] := B \bowtie C$
2. $X[y,z,q,y',z',q'] := X \ltimes V$
3. ...

First step yields $||D||^2$-factor since the big bag is covered by 2 atoms B and C



Treewidth revisited

Definition (Generalised hypertreewidth [Gottlob et al., 2003])

A generalised hypertree decomposition of a hypergraph $H = (V, E)$ is a triple (B, E_T, λ), where

1. (B, E_T) is a tree decomposition of H.
2. λ is a mapping that assigns a subset of E to each bag $b \in B$.
3. For each bag $b \in B$, $b \subseteq \bigcup_{e \in \lambda(b)} e$.

The width of a hypertree is the cardinality of its largest λ-label, i.e., $\max_{b \in B} |\lambda(b)|$.
The generalised hypertreewidth of H is the minimum over over widths of generalised hypertree decompositions of H.

CQ-Evaluation for fixed generalised hypertreewidth

Theorem

Fix $k \geq 1$. Then CQ-Evaluation, restricted to queries of generalised hypertreewidth at most k, can be solved in time

$$2^{||q||^c} + O(||D||^k \cdot ||q||), \quad \textit{for some integer } c \geq 1.$$

Key pointers:

- Finding generalised hypertree decompositions is generally hard. $2^{||q||^c}$ accounts for what is essentially a brute-force construction.
- Generalised hypertreewidth = largest bag cover $\implies$ $||D||^k$-factor

Generalised hypertreewidth can be further generalised with the notion a fractional hypertreewidth [Grohe and Marx, 2006], but let's move on...

Introduction

Complexity of CQ-Evaluation

Joins with Information Theory

Conclusion

Efficient CQ-Evaluation: Recap

Recall CQ-Evaluation has two-partite input: (q, D)

Structural properties of q:

- Treewidth
- Generalised hypertreewidth

...lead to efficient query evaluation.

Properties of D?

- Cardinalities of relations?
- Constraints on relations?

Can we combine information from q **and** D to obtain efficient algorithms?

Target

Let us consider again the case where the CQ q is non-Boolean, i.e., the output $q(D)$ is a relation

Target: Devise efficient algorithms for computing $q(D)$

Strategy:

1. Given structural properties of q and information about D, determine the worst-case size of the output $q(D)$.
2. Devise algorithms that run in time proportional to this worst-case output size.

Join queries

We will consider join queries q.

The (natural) join of R_1 and R_2, denoted $R_1 \bowtie R_2$, is the set of tuples that are formed by combining tuples from R_1 and R_2 which agree on their common attributes.
"Formally":

$$R \bowtie S = \{t \mid t[\text{att}(R)] \in R, t[\text{att}(S)] \in S\},$$

where $t[A]$ is the projection of a tuple t on an attribute set A.

Note: Join is associative and commutative

3-way join example

Example

Input Tables:

$R[A, B] =$

A	B
1	10
2	20
3	30

$S[A, C] =$

A	C
1	100
2	200
4	400

$T[B, C] =$

B	C
10	100
20	300
30	300

Result of the Join:

$R[A, B] \bowtie S[A, C] \bowtie T[B, C] =$

A	B	C
1	10	100

Joins vs. CQs

A join query $R_1 \bowtie \cdots \bowtie R_n$ can be viewed as a CQ of the form

$$\text{Answer}(\vec{x}) \coloneq R_1(\vec{y}_1), \ldots, R_n(\vec{y}_n)$$

in which:

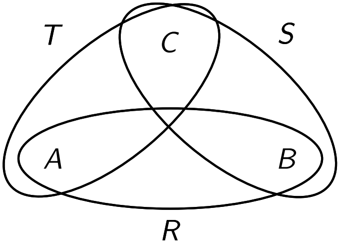
1. Each relation name R_i occurs exactly once.
2. For every $i \in [n]$, no variables are repeated in $\vec{y}_i$.
3. Every variable in $\vec{y}_i$ also appears in $\vec{x}$.

Example: "Triangle Query" $q_\triangle$

Consider the join query:

$$q_\triangle = R[A, B] \bowtie S[B, C] \bowtie T[C, A]$$

The hypergraph of $q_\triangle$ visualizes its structure:



Question: How many tuples can there be in $q_\triangle(D)$, where D is a database?

Evaluating $q_\triangle(D)$

Assume R, S, T each have N tuples

Trivially, $|q_\triangle(D)| \leq N^3$ (size of Cartesian product $|R| \cdot |S| \cdot |T|$)

However, $|q_\triangle(D)| \leq N^2$ due to attribute constraints. Consider the following computation:

1. Compute $R \bowtie S$, selecting tuples matching on B.
   - Output size at most $|R \times S| \leq N^2$
2. Join the result with T, selecting tuples matching on A and C.
   - Can only remove tuples from the previous join
3. $\implies$ Final output size at most $|R \times S| \leq N^2$.

Our naïve analysis yields:

$$|q_{\triangle}(D)| \leq N^2$$

The so-called AGM bound [Atserias et al., 2013] yields:

$$|q_{\triangle}(D)| \leq N^{3/2} \qquad (1)$$

Next: Derivation of (1)

Entropy

(Shannon) entropy of a random variable X with a finite domain D and a probability mass function p:

$$H(X) := -\sum_{x \in D} p(x) \log p(x).$$

Useful bounds: $0 \leq H(X) \leq \log |D|$

Entropic function $h(\boldsymbol{X}_\alpha) := H(\boldsymbol{X}_\alpha)$ $(\alpha \subseteq [n])$ for Shannon entropies H arising from marginals of some joint distribution of random variables $X_1, \ldots, X_n$

Entropic region Γ_n^*: the subset of $\mathbb{R}^{2^n}$ consisting of the entropic functions/vectors over n random variables

Laws of information

Example: conditioning decreases entropy

$$h(\boldsymbol{X} \mid \boldsymbol{Y}) = h(\boldsymbol{X}\boldsymbol{Y}) - h(\boldsymbol{Y}) \leq h(\boldsymbol{X})$$

Polymatroid axioms:

- $h(\emptyset) = 0$
- $h(\boldsymbol{X}) \leq h(\boldsymbol{X}\boldsymbol{Y})$ (monotonicity)
- $h(\boldsymbol{X}) + h(\boldsymbol{XYZ}) \leq h(\boldsymbol{X}\boldsymbol{Y}) + h(\boldsymbol{X}\boldsymbol{Z})$ (submodularity)

Polymatroid axioms sound but incomplete [Zhang and Yeung, 1998]; there is no finite axiomatic characterisation for entropic functions [Matús, 2007]

Derivation of AGM bound I

Instance of **Shearer's Lemma**:

$$
\begin{aligned}
& \tfrac{1}{2}\big(h(XY) + && h(XZ) + && h(YZ)\big) \\
= \; & \tfrac{1}{2}\big(h(X) + h(Y \mid X) + && h(X) + h(Z \mid X) + && h(Y) + h(Z \mid Y)\big) \\
\geq \; & \tfrac{1}{2}\big(h(X) + h(Y \mid X) + && h(X) + h(Z \mid XY) + && h(Y \mid X) + h(Z \mid XY)\big) \\
= \; & h(X) + h(Y \mid X) + h(Z \mid XY) \\
= \; & h(XYZ)
\end{aligned}
$$

Derivation of AGM bound II

Consider $q_\triangle = R[A, B] \bowtie S[B, C] \bowtie T[C, A]$ and a database $D = \{R, S, T\}$ where each relation at most of size N

If h is the entropic function of the output $q_\triangle(D)$ uniformly distributed:

$$\begin{aligned}\log|q_\triangle(D)| = h(ABC) &\leq \frac{1}{2}\big(h(AB) + h(AC) + h(BC)\big)\\ &\leq \frac{1}{2}\big(\log|R| + \log|S| + \log|T|\big)\\ &\leq \frac{3}{2}\log N\end{aligned}$$

Fractional Edge Cover

Definition

A fractional edge cover of a hypergraph $H = (V, E)$ is a function

$$f : E \to \mathbb{Q}_{\geq 0}$$

such that, for each node $v \in V$, it holds that

$$\sum_{v \in e} f(e) \geq 1.$$

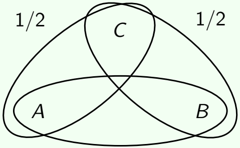
The cover f is called minimal when it minimises the weight

$$\sum_{e \in E} f(e).$$

Fractional Edge Cover for Triangle query

Example

Let $f(e) = 1/2$ for all $e \in E$. Then,
$f(e) \geq 0$ for all hyperedges e, and:



$$\sum_{A \in e} f(e) = f(\{A, B\}) + f(\{A, C\}) = 1/2 + 1/2 \geq 1$$

$$\sum_{B \in e} f(e) = f(\{A, B\}) + f(\{B, C\}) = 1/2 + 1/2 \geq 1$$

$$\sum_{C \in e} f(e) = f(\{A, C\}) + f(\{B, C\}) = 1/2 + 1/2 \geq 1$$

Minimality of f

A minimal fractional edge cover for $q_\triangle = R[A, B] \bowtie S[B, C] \bowtie T[C, A]$ can be found by solving the following linear program:

$$\begin{aligned} \text{minimize} \quad & x_R + x_S + x_T \\ \text{subject to} \quad & x_R + x_T \geq 1, \\ & x_R + x_S \geq 1, \\ & x_S + x_T \geq 1, \\ \text{and} \quad & x_R \geq 0, \quad x_S \geq 0, \quad x_T \geq 0. \end{aligned}$$

minimal fractional edge cover = values of x_R, x_S, x_T at the optimal solution

AGM Bound for Join Queries

Theorem (AGM Bound)

Consider a join query $q = R_1 \bowtie \cdots \bowtie R_n$ over schema $\boldsymbol{S}$ and a fractional edge cover f of q. Then, for every database D, we have:

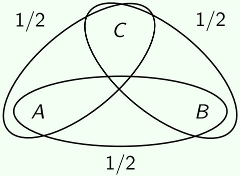
$$|q(D)| \leq \prod_{i=1}^{n} |R_i|^{f(\boldsymbol{S}(R_i))}. \tag{2}$$

If f is a minimal, there are arbitrarily large databases D for which Eq. (2) is an equality,

Optimal AGM bound for $q_\triangle$

Example

Assume $|R| = |S| = |T| = N$. Then:



$$|q_\triangle(D)| \leq \prod_{i=1}^{n} |R_i|^{f(\mathbf{S}(R_i))} = |R|^{1/2} \cdot |S|^{1/2} \cdot |T|^{1/2} = \sqrt{|R| \cdot |S| \cdot |T|} = N\sqrt{N}$$

Inoptimality of binary joins

Suppose each of R, S, T has the form

1	1
1	2
⋮	⋮
1	$(N+1)/2$
2	1
⋮	⋮
$(N+1)/2$	1

} N tuples

- AGM bound $= N\sqrt{N}$
- Any intermediate binary join $R \bowtie S, R \bowtie T, S \bowtie T$ contains more than $(N/2)^2 = \frac{1}{4}N^2$ tuples

Q: Possible to compute $R \bowtie S \bowtie T$ in $O(N\sqrt{N})$?

Attribute-Elimination Join for $R[A, B] \bowtie S[B, C] \bowtie T[A, C]$

1. Compute $L_1 := \pi_A(R \bowtie T)$.
2. For each $a \in L_1$:
   - Compute values $b \in \pi_B(R \bowtie S)$ s.t. $(a, b) \in R$ and $(b, c) \in S$.
   - Add pairs (a, b) to L_2.
3. For each $(a, b) \in L_2$:
   - Compute values $c \in \pi_C(S \bowtie T)$ s.t. $(b, c) \in S$ and $(c, a) \in T$.
   - Add triples (a, b, c) to L_3.
4. Return L_3.

Relations R, S, T:

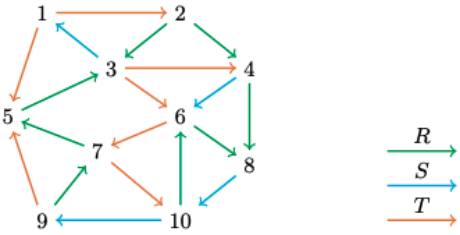


Fig. source: [Arenas et al., 2022]

We obtain:

- $L_1 = \{5, 2, 6, 7, 4, 10\}$
- $L_2 = \{(5, 3), (2, 3), (2, 4), (6, 8), (4, 8)\}$
- $L_3 = \{(5, 3, 1), (2, 3, 1)\}$

Complexity of AEJoin

Theorem

Consider a join query $q = R_1 \bowtie \ldots \bowtie R_n$ *over attributes* $A_1, \ldots, A_m$*. Then the Attribute-Elimination Join algorithm computes the output in time* $\tilde{O}\left(n \cdot m \cdot \prod_{j=1}^{n} |R_j|^{x_j}\right)$*, where* $(x_1, \ldots, x_n)$ *is a fractional edge cover of* q*.*

Note: For a function $f(\vec{x})$, we write $\tilde{O}(f(\vec{x})) = O(f(\vec{x}) \log f(\vec{x}))$

A join algorithm with running time $\tilde{O}(nm|q(D)|)$ is called a worst-case optimal join algorithm.

Efficient CQ-Evaluation: Recap 2

CQ-Evaluation has two-partite input: (q, D). We considered:

Structural properties of q:

- **Fractional edge cover**

Properties of D:

- **Cardinalities of relations**

...to sketch a worst-case optimal join algorithm

Efficient CQ-Evaluation: Recap 2

CQ-Evaluation has two-partite input: (q, D). We considered:

Structural properties of q:

- **Fractional edge cover**

Properties of D:

- **Cardinalities of relations**
- Constraints on relations?

...to sketch a worst-case optimal join algorithm

More input information: constraints

Target: Estimate $|q(D)|$ given

1. join query $q = R_1[\boldsymbol{X}_1] \bowtie \ldots \bowtie R_n[\boldsymbol{X}_n]$
2. degree constraints w.r.t. bounds $\boldsymbol{B}$

Degree constraints

Degree $\deg_R(\boldsymbol{V} \mid \boldsymbol{U} = \boldsymbol{u})$: number of distinct values of $\boldsymbol{V}$ in R under $\boldsymbol{U} = \boldsymbol{u}$

Max-degree $\deg_R(\boldsymbol{V} \mid \boldsymbol{U})$: maximum of degrees $\deg_R(\boldsymbol{V} \mid \boldsymbol{U} = \boldsymbol{u})$ over $\boldsymbol{u}$

- Functional dependencies $\boldsymbol{U} \to \boldsymbol{V}$ definable by $\deg_R(\boldsymbol{V} \mid \boldsymbol{U}) \leq 1$
- Size bounds $|R| \leq B$ definable by $\deg_R(\boldsymbol{U} \mid \emptyset) \leq B$

Degree statistics: Set Σ of conditionals $(\boldsymbol{V} \mid \boldsymbol{U})$. A conditional $(\boldsymbol{V} \mid \boldsymbol{U})$ is guarded by a relation $R[\boldsymbol{X}]$ if $\boldsymbol{U}\boldsymbol{V} \subseteq \boldsymbol{X}$.

Entropic bound

Target: Estimate $|q(D)|$ given

1. join query $q = R_1[\boldsymbol{X}_1] \bowtie \ldots \bowtie R_n[\boldsymbol{X}_n]$
2. degree statistics Σ and size vector $\boldsymbol{B} = (B_\sigma)_{\sigma\in\Sigma}$ where each $\sigma \in \Sigma$ guarded by some atom $R_\sigma(\boldsymbol{X}_\sigma)$
3. $D \models (\Sigma, \boldsymbol{B})$, meaning $\deg_{R_\sigma}(\boldsymbol{V} \mid \boldsymbol{U}) \leq B_\sigma$ for all $\sigma = (\boldsymbol{V} \mid \boldsymbol{U}) \in \Sigma$

Entropic bound (w.r.t. degree constraints) [Khamis et al., 2017] defined by[2]

$$Ent(q, \boldsymbol{B}, \Sigma) = \sup_{\boldsymbol{w}:\Gamma_n^* \models (3)} \prod_{\sigma\in\Sigma} B_\sigma^{w_\sigma},$$

where

$$\sum_{\sigma\in\Sigma} w_\sigma h(\sigma) \geq h(\boldsymbol{X}) \tag{3}$$

[2] $\Gamma_n^* \models (3)$ denotes that (3) holds for all functions $h \in \Gamma_n^*$ (i.e., all entropic functions h)

Derivation of entropic bound

Theorem

Let $q(\boldsymbol{X})$ be a join query and D a database such that each $\sigma = (\boldsymbol{V} \mid \boldsymbol{U}) \in \Sigma$ is guarded by $R_\sigma[\boldsymbol{X}] \in q$ s.t. $\deg_{R_\sigma^D}(\boldsymbol{V} \mid \boldsymbol{U}) \leq B_\sigma$. If $\Gamma_n^* \models \sum_{\sigma\in\Sigma} w_\sigma h(\sigma) \geq h(\boldsymbol{X})$, then

$$|q(D)| \leq \prod_{\sigma\in\Sigma} B_\sigma^{w_\sigma}$$

Proof.

If h is the entropic function of $q(D)$ uniformly distributed, then

$$h(\sigma) = \mathbb{E}_{\boldsymbol{u}}[h(\boldsymbol{V} \mid \boldsymbol{U} = \boldsymbol{u})] \leq \max_{\boldsymbol{u}} h(\boldsymbol{V} \mid \boldsymbol{U} = \boldsymbol{u}) \leq \max_{\boldsymbol{u}} \log \deg_{q(D)}(\boldsymbol{V} \mid \boldsymbol{U} = \boldsymbol{u})$$
$$\leq \max_{\boldsymbol{u}} \log \deg_{R_\sigma}(\boldsymbol{V} \mid \boldsymbol{U} = \boldsymbol{u}) = \log \deg_{R_\sigma}(\boldsymbol{V} \mid \boldsymbol{U}) \leq \log B_\sigma$$

whence $\log|q(D)| = h(\boldsymbol{X}) \geq \sum_{\sigma\in\Sigma} w_\sigma h(\sigma) \geq \sum_{\sigma\in\Sigma} w_\sigma \log B_\sigma$. □

Computability

The entropic bound

$$Ent(q, \boldsymbol{B}, \Sigma) = \sup_{\boldsymbol{w}:\Gamma_n^* \models \phi} \prod_{\sigma \in \Sigma} B_\sigma^{w_\sigma}, \text{ where } \phi = \sum_{\sigma \in \Sigma} w_\sigma h(\sigma) \geq h(\boldsymbol{X}),$$

- Asymptotically tight
- Not known to be computable.
- Polynomial-time computable if each $\sigma = (\boldsymbol{V} \mid \boldsymbol{U}) \in \Sigma$ s.t. $\boldsymbol{U}$ is a singleton [Im et al. 2022; H. 2024].

Polymatroid bound (obtained by replacing Γ_n^* with Γ_n) not tight but computable in exponential time in n.

Recap: More input information

Target: Estimate $|q(D)|$ given

1. join query $q = R_1[\boldsymbol{X}_1] \bowtie \ldots \bowtie R_n[\boldsymbol{X}_n]$
2. degree constraints w.r.t. bounds $\boldsymbol{B}$

Things get more complicated when incorporating information about constraints. See [Suciu, 2023] for an in-depth review of the applications of information theory in databases.

Introduction

Complexity of CQ-Evaluation

Joins with Information Theory

Conclusion

CQ-Evaluation problem (given a database D and a Boolean CQ q, is q true for D?)

- Generally NP-complete
- Tractable when the hypergraph of the query is nearly acyclic (treewidth, generalised hypertreewidth)

Join computation (outputs of non-Boolean CQs)

- Leverage structural properties of q (fractional edge cover) and cardinalities of relations in D
- Worst-case optimal join algorithms run in time proportional to the worst-case output size, the number of attributes, and the number of relations.

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